Lecture 7

Goals:

- Signals as Vectors, Noise as Vectors
- Optimum Detection in AWGN

Decomposition of Signal and Noise

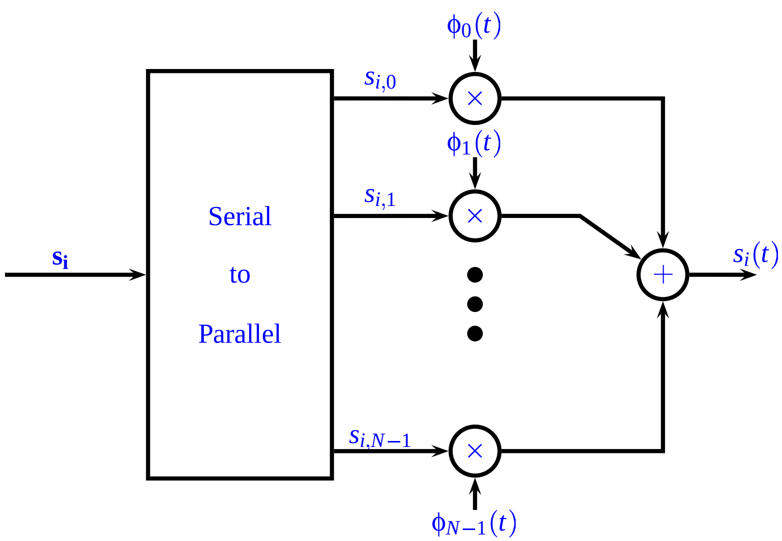
Given a set of signals $s_0(t),\ldots,s_{M-1}(t)$ there exists a set of orthonormal signals $\phi_0(t),\phi_1(t),\ldots,\phi_{N-1}(t)$ with $N \le M$ such that

$$s_i(t) = \sum_{m=0}^{N-1} s_{i,m}\phi_m(t)$$

For any complete orthonormal set of signals $\phi_0(t),\phi_1(t),\ldots$ we can represent a noise process as random variables and deterministic orthonormal functions

$$n(t) = \sum_{m=0}^{\infty} n_m\phi_m(t)$$

Signal



Decomposition of Signal and Noise

Consider a communication system that transmits one of M signals. $s_0(t),\ldots,s_{M-1}(t)$ in additive white Gaussian noise. s Then given $s_i(t)$ was transmitted the received signal is

$$\begin{aligned} r(t) &= s_i(t) + n(t) \\ &= \sum_{m=0}^{\infty}(s_{i,m}+n_m)\phi_m(t) \end{aligned}$$

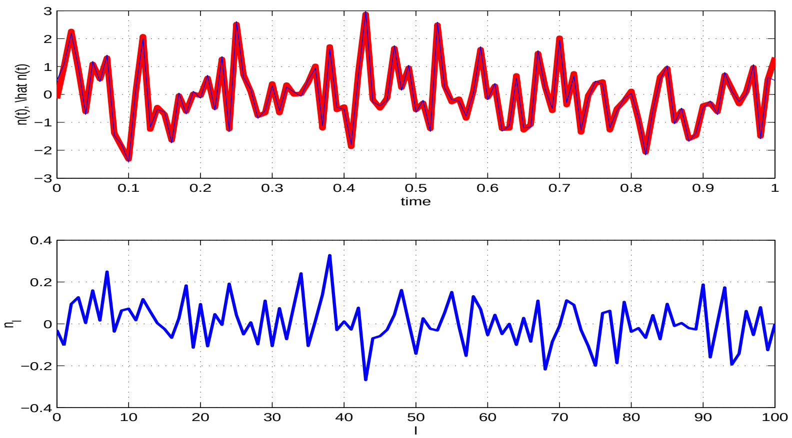
Define $r_m = s_{i,m} + n_m$. Then

$$r(t) = \sum_{m=0}^{\infty} r_m\phi_m(t)$$

We can determine the (random) variable r_m by

$$r_m = \int r(t)\phi_m^*(t)dt$$

Decomposition of Noise



Example

$$\begin{aligned} s_0(t) &= Ap_T(t) \\ s_1(t) &= -Ap_T(t) \end{aligned}$$

Let $\phi_l(t) = \sqrt{\frac{1}{T}}\exp\{j2\pi l f_0 t\}p_T(t)$ where $f_0 = 1/T$. Then

$$\begin{aligned} s_0(t) &= \sqrt{E}\phi_0(t) \\ s_1(t) &= -\sqrt{E}\phi_0(t) \\ n(t) &= \sum_{m=0}^{\infty} n_m\phi_m(t) \\ r(t) &= \sum_{m=0}^{\infty} r_m\phi_m(t) \end{aligned}$$

$$\begin{aligned} r_m &= \int r(t)\phi_m^*(t)dt \\ &= \int (s_i(t)+n(t))\phi_m^*(t)dt \\ &= s_{i,m}+n_m \end{aligned}$$

Note that we can recover completely $r(t)$ if we know the coefficients $r_m, m = 0,1,....$ So the optimal decision based on observing $r_0, r_1, ...$ is also the optimal decision based on observing $r(t)$. Given signal $s_i(t)$ is transmitted we can determine the probability density of r_m as follows. First, r_m is Gaussian since it is the result of integrating Gaussian noise. Second the mean of r_m is $s_{i,m}$ and the variance is $N_0/2$. So the probability density of r_m conditioned on signal i transmitted (event H_i) is

$$\begin{aligned} p_i(r_m) &= f_{r_m|H_i}(r_m) \\ &= \frac{1}{\sqrt{2\pi}\sqrt{N_0/2}}\exp\{-\frac{(r_m - s_{i,m})^2}{2(N_0/2)}\} \end{aligned}$$

Next note that r_m is independent of r_m for $m \neq n$. Thus

$$f_{r_0,r_1,\ldots,r_k|H_i}(x_0,x_1,x_2,\ldots,x_k) = \prod_{m=0}^{k} f_{r_m|H_i}(x_m) = \prod_{m=0}^{k} p_i(x_m)$$

M-ary Detection Problem

Consider the problem of deciding which of M hypothesis is true based on observing a random variable (vector) r. The performance criteria we consider is the average error probability. That is the probability of deciding anything except hypothesis H_j when hypothesis H_j is true.

The underlying model is that there is a conditional probability density (mass) function of the observation r given each hypothesis H_j. There are disjoint decision regions $R_0, R_1, \ldots, R_{M-1}$. When $r \in R_m$ the receiver decides H_m.

$$E[P_e] = \sum_{i=0}^{M-1} P_{e,i}\pi_i = \sum_{i=0}^{M-1} P\{\text{don't decide } H_i|H_i\}\pi_i = \sum_{i=0}^{M-1} P\{r \in \bigcup_{l=0,l\neq i}^{M-1} R_l|H_i\}\pi_i$$

$$= \sum_{i=0}^{M-1}\left[\sum_{l\neq i} P\{\text{decide } H_l|H_i \text{ true}\}\right]\pi_i$$
$$= \sum_{i=0}^{M-1} [1 - P\{\text{decide } H_i|H_i \text{ true}\}]\pi_i$$
$$= \sum_{i=0}^{M-1}\pi_i - \sum_{i=0}^{M-1}\int_{R_i} p_i(r)\pi_i dr$$
$$= 1 - \sum_{i=0}^{M-1}\int_{R_i} p_i(r)\pi_i dr \quad .$$

The decision rule that minimizes average probability of error assigns r to R_i if $p_i(r)\pi_i = \max_{0\leq j\leq M-1} p_j(r)\pi_j$. Let $p(r)$ be an arbitrary density function that is nonzero everywhere $p_i(r)$ is nonzero then an equivalent decision rule is to assign r to R_i if

$$\frac{p_i(r)}{p(r)}\pi_i = \max_{0\leq j\leq M-1}\frac{p_j(r)}{p(r)}\pi_j.$$

Thus for M hypotheses the decision rule that minimizes average error probability is to choose i so that $p_i(r)\pi_i > p_j(r)\pi_j, \ \forall \ j \neq i$. Let

$$\Lambda_{i,j} = \frac{p_i(r)}{p_j(r)}$$

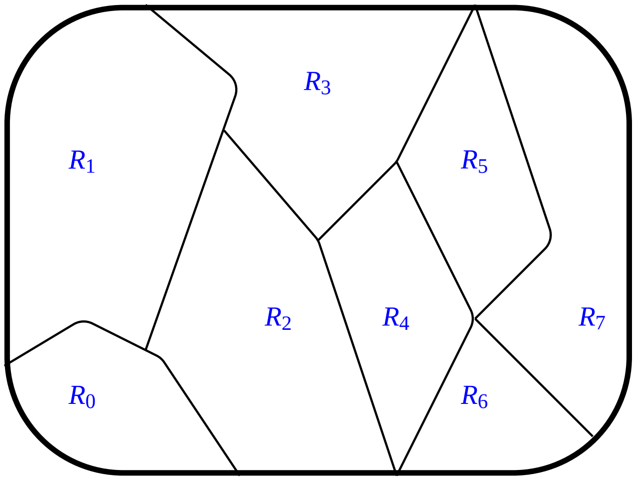
where $i = 0,1,\ldots,M-1$, $j = 0,1,\ldots,M-1$. Then the optimal decision rule is:

$$\text{Choose } i \text{ if } \Lambda_{i,j} > \frac{\pi_j}{\pi_i} \text{ for all } j \neq i.$$

We will usually assume $\pi_i = \frac{1}{M}\ \forall i$. (If not we should do source encoding to reduce the entropy (rate)). For this case the optimal decision rule is

$$\text{Choose } i \text{ if} \quad \Lambda_{i,j} > 1 \ \ \forall \ \ j \neq i.$$

Decision Regions



Example 1: Additive White Gaussian Noise

Consider three signals in additive white Gaussian noise. For additive white Gaussian noise $K(s,t) = \frac{N_0}{2}\delta(t-s)$. Let $\{\varphi_i(t)\}_{i=0}^{\infty}$ be any complete orthonormal set on $[0,T]$. Consider the case of 3 signals. Find the decision rule to minimize average error probability. First expand the noise using orthonormal set of functions and random variables.

$$n(t) = \sum_{i=0}^{\infty} n_i \varphi_i(t)$$

where $E[n_i] = 0$ and $\text{Var}[n_i] = N_0/2$ and $\{n_i\}_{i=0}^{\infty}$ is an independent identically distributed (i.i.d.) sequence of random variables with Gaussian density functions.

Let

$$s_0(t) = \varphi_0(t) + 2\varphi_1(t)$$

$$s_1(t) = 2\varphi_0(t) + \varphi_1(t)$$
$$s_2(t) = \varphi_0(t) - 2\varphi_1(t)$$

Note that the energy of each of the three signals is the same, i.e. $\int_0^T s_i^2(t)dt = ||s_i||^2 = 5$. Then we have a three hypothesis testing problem.

$$H_0 : r(t) = s_0(t) + n(t) = \sum_{i=0}^{\infty}(s_{0,i} + n_i)\varphi_i(t)$$
$$H_1 : r(t) = s_1(t) + n(t) = \sum_{i=0}^{\infty}(s_{1,i} + n_i)\varphi_i(t)$$
$$H_2 : r(t) = s_2(t) + n(t) = \sum_{i=0}^{\infty}(s_{2,i} + n_i)\varphi_i(t)$$

The decision rule to minimize the average error probability is given as follows

$$\text{Decide } H_i \text{ if } \pi_i p_i(\mathbf{r}) = \max_j \pi_j p_j(\mathbf{r})$$

First let us consider the first $L+1$ variables and normalize each side by the

density function for the noise alone. The noise density function for $L+1$ variables is

$$p^{(L)}(\mathbf{r}) = \left(\frac{1}{\sqrt{2\pi N_0/2}}\right)^N \exp\{-\frac{1}{2\frac{N_0}{2}}\sum_{m=0}^{L} r_m^2\}$$

The the optimal decision rule is equivalent to

$$\text{Decide } H_i \text{ if } \pi_i \frac{p_i(\mathbf{r})}{p(\mathbf{r})} = \max_j \pi_j \frac{p_j(\mathbf{r})}{p(\mathbf{r})}.$$

As usual assume $\pi_i = 1/M$. Then

$$\frac{p_0^{(L)}(\mathbf{r})}{p^{(L)}(\mathbf{r})} = \frac{\left(\frac{1}{\sqrt{2\pi N_0/2}}\right)^L \exp\{-\frac{1}{2\frac{N_0}{2}}[\sum_{i=0,1}(r_i - s_{0,i})^2 + \sum_{i=2}^{L} r_i^2]\}}{\left(\frac{1}{\sqrt{2\pi N_0/2}}\right)^L \exp\{-\frac{1}{2\frac{N_0}{2}}\sum_{i=0}^{1} r_i^2 + \sum_{i=2}^{L}\}}$$
$$= \exp\{-\frac{1}{N_0}[\sum_{i=0,1}(r_i - s_{0,i})^2 - r_i^2]\}$$

$$= \quad \exp\{+\frac{1}{N_0}[2r_1 + 4r_2 - 5]\}.$$

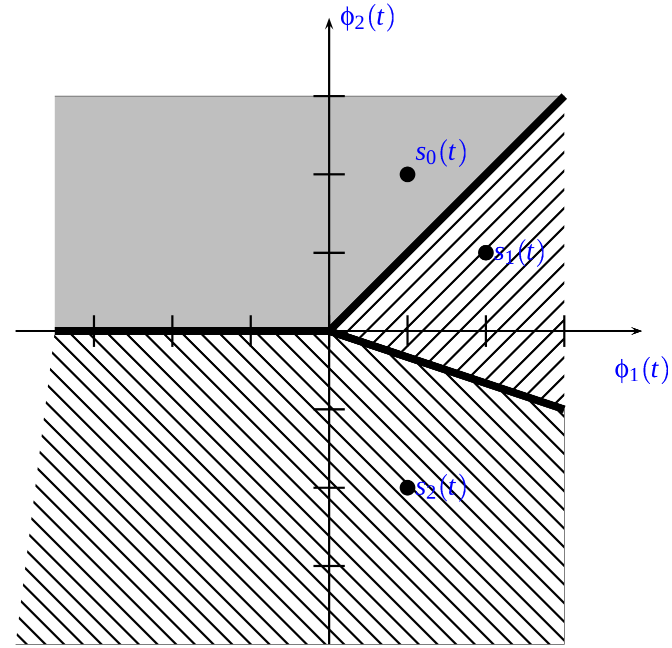
Now since the above doesn't depend on L we can let $L \to \infty$ and the result is the same, i.e.

$$\frac{p_0(\mathbf{r})}{p(\mathbf{r})} \triangleq \lim_{L\to\infty} \frac{p_0^{(L)}(\mathbf{r})}{p^{(L)}(\mathbf{r})} = \exp\{+\frac{1}{N_0}[2r_0 + 4r_1 - 5]\}.$$

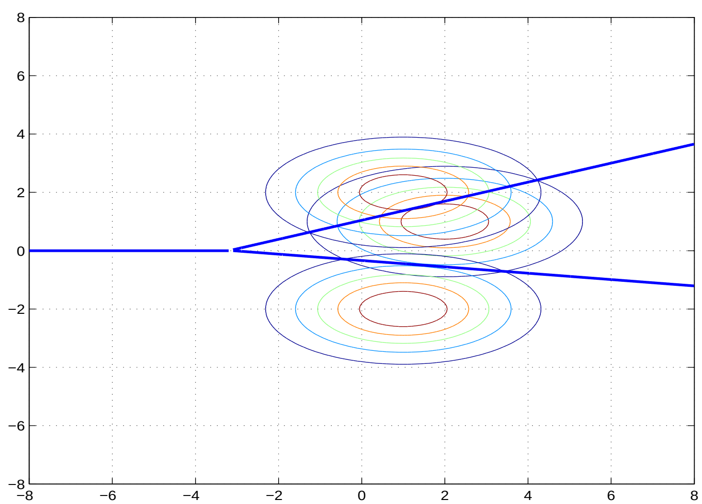
Similarly

$$\frac{p_1(\mathbf{r})}{p(\mathbf{r})} = \exp\{+\frac{1}{N_0}[4r_0 + 2r_1 - 5]\}$$

$$\frac{p_2(\mathbf{r})}{p(\mathbf{r})} = \exp\{+\frac{1}{N_0}[2r_0 - 4r_1 - 5]\}.$$



Decision Regions



Likelihood Ratio for Real Signals in AWGN

Assume two signals in Gaussian noise.

$$H_0 \quad : \quad r(t) = s_0(t) + n(t)$$
$$H_1 \quad : \quad r(t) = s_1(t) + n(t)$$

Goal: Find decision rule to minimize the average error probability.

Let $n(t)$ autocorrelation function $R((s,t) = \frac{N_0}{2}\delta(t-s)$ We assume that $n(t)$ is a zero mean white Gaussian noise random process.

Karhunen-Loeve Expansion

By Karhunen-Loeve expansion

$$n(t) = \sum_{m=0}^{\infty} n_m \varphi_i(t)$$

where n_i are Gaussian random variables with mean 0 variance $\frac{N_0}{2}$ and $E[n_m n_k] = 0,\ m \neq k$. Thus n_m and n_k are independent. Since $\{\varphi_m(t); m = 0, 1, \ldots\}$ is a complete orthonormal set and we assume $s_j(t)$ has finite energy we have

$$s_j(t) = \sum_{m=0}^{\infty} s_{j,m}\varphi_m(t) = \sum_{m=0}^{N-1} s_{j,m}\varphi_m(t).$$

This last equality is because we only need a finite ($N \leq M$) orthonormal waveforms to represent a set of M signals. Equivalently $s_{j,i} = 0$ for $i \geq N$.

Thus

$$H_j : r(t) = \sum_{m=0}^{\infty} (s_{j,m} + n_m)\varphi_m(t)$$

$$r_m = s_{j,m} + n_m, \quad m = 0, 1, 2, \ldots$$

Define

$$\Lambda_{j,i}(L) \triangleq \frac{p_j(r_0, r_1, \ldots, r_L)}{p_i(r_0, r_1, \ldots, r_L)} .$$

$$\Lambda_{j,i}(r(t)) \triangleq \lim_{L\to\infty} \Lambda_{j,i}(L)$$

where r_m is Gaussian with mean $s_{j,m}$ variance $N_0/2$.

$$p_j(r_m) = \frac{1}{\sqrt{N_0\pi}} \exp\left\{-\frac{1}{N_0}(r_m - s_{j,m})^2\right\}$$

$$p_j(\underline{r}) = \prod_{m=0}^{L} p_j(r_m) = \prod_{m=0}^{L} (\sqrt{N_0\pi})^{-1} \exp\left\{-\frac{1}{N_0}\sum_{m=0}^{L}(r_m - s_{j,m})^2\right\}$$

$$\Lambda_{j,l}(L) = \frac{p_j^L(\underline{r})}{p_l^L(\underline{r})} = \frac{\prod_{m=0}^{L} (\sqrt{N_0\pi})^{-1}}{\prod_{m=0}^{L} (\sqrt{N_0\pi})^{-1}} \frac{\exp\left\{-\frac{1}{N_0}\sum_{m=0}^{L}(r_m - s_{j,m})^2\right\}}{\exp\left\{-\frac{1}{N_0}\sum_{m=0}^{L}(r_m - s_{l,m})^2\right\}}$$

$$= \exp\left\{-\frac{1}{N_0}\sum_{m=0}^{L}[r_m^2 - 2r_m s_{j,m} + s_{j,m}^2 - r_m^2 + 2r_m s_{l,m} - s_{l,m}^2]\right\}$$

$$= \exp\left\{-\frac{1}{N_0}\sum_{m=0}^{L}[s_{j,m}^2 - s_{l,m}^2 + 2r_i(s_{l,m} - s_{j,m})]\right\}.$$

If we take the limit as $L \to \infty$ we get

$$\Lambda_{j,l}(r(t)) = \exp\left\{-\frac{1}{N_0}(E_0 - E_1 + 2(r, s_l - s_j)]\right\}.$$

$$\Lambda_{j,l}(r(t)) = \exp\left\{-\frac{1}{N_0}[(s_j, s_j) - (s_l, s_l) + 2(r, s_l) - 2(r, s_j)]\right\}.$$

or equivalently

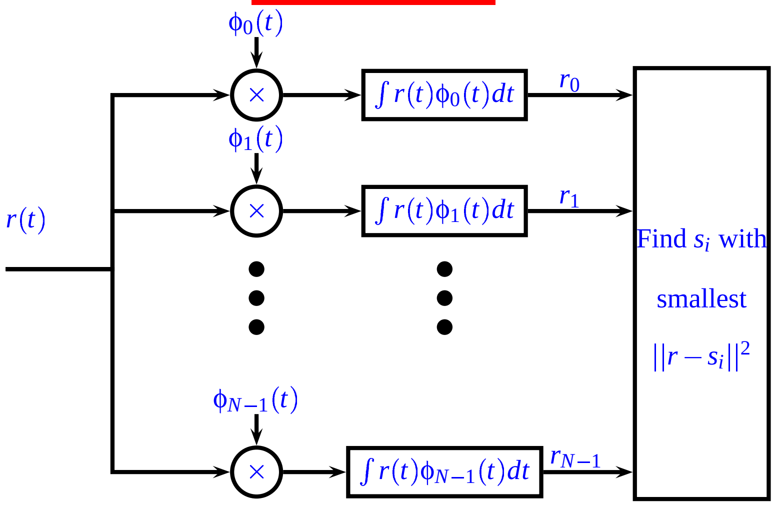
$$\Lambda_{j,l}(r(t)) = \exp\left\{-\frac{1}{N_0}[||s_j||^2 - ||s_l||^2 + 2(r, s_l - s_j)]\right\}$$

$$= \exp\left\{-\frac{1}{N_0}[||r - s_j||^2 - ||r - s_l||^2]\right\}$$

The optimum decision rule for additive white Gaussian noise is then to choose i if

$$||s_i - r||^2 = \min_{0 \leq j \leq M-1} ||s_j - r||^2.$$

Demodulator



Example: M equal energy signals

Now consider the optimum receiver for M-ary equally likely signals and the associated error probability. Assume the M signals are equienergy signals and equiprobable. The decision rule derived previously for AWGN in this case simplifies to

$$\text{Decide } H_i \text{ if } ||s_i - r||^2 = \min_{0\le j\le M-1} ||s_j - r||^2.$$

Now since the M signals are equienergy we can write this as

$$||s_j - r||^2 = ||s_j||^2 - 2(s_j, r) + ||r||^2.$$

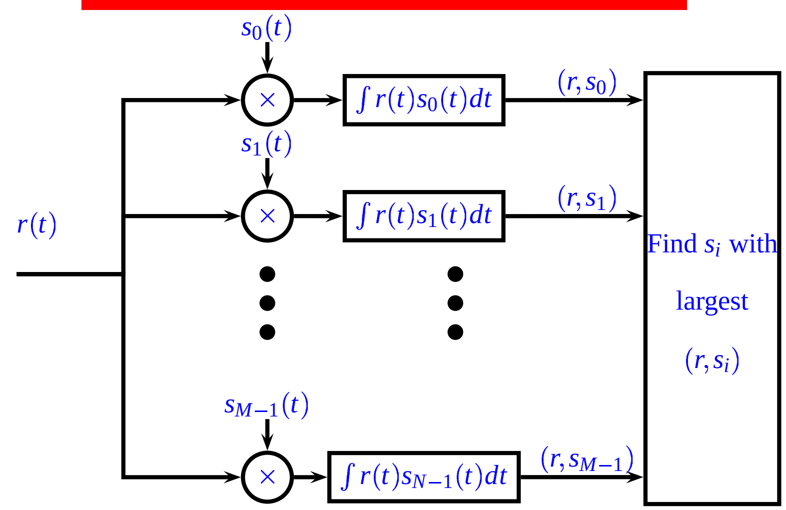
The first term above is constant for each j as is the last term. Thus finding the minimum is equivalent to finding the maximum of

$$(s_j, r).$$

Thus the receiver should compute the inner product between the M different

signals and find the largest such correlation. If the signals are all of duration T, i.e. zero outside the interval $[0,T]$ then this is also equivalent to filtering the received signal with a filter with impulse response $s_j(T-t)$, sampling the output of the filter at time T and choosing the largest.

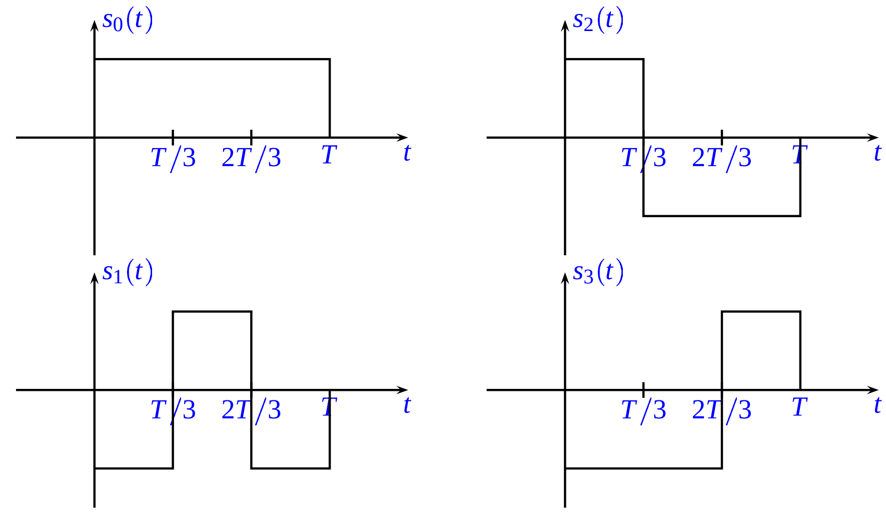
Demodulator (Equal Energy Case)



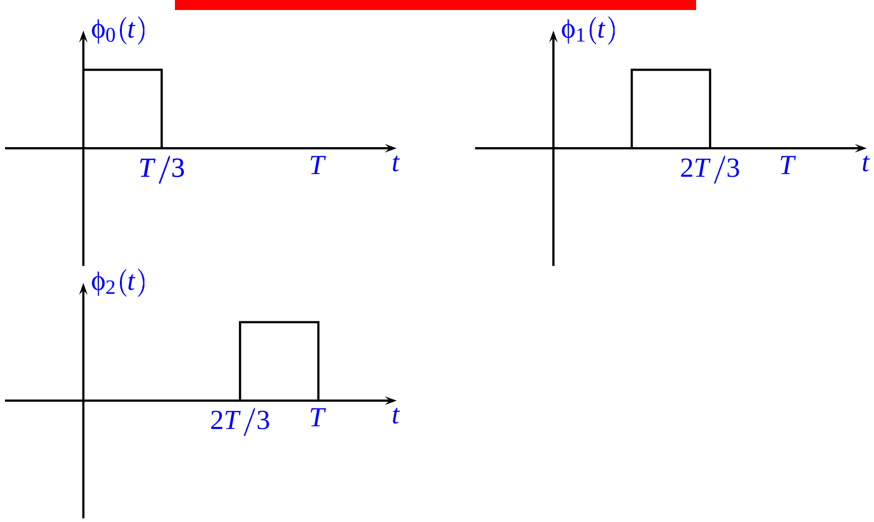
Notes about Optimum Receiver in AWGN

- Consider the case of equally likely signals ($\pi_0 = \ldots = \pi_{M-1} = 1/M$).
- The optimum receiver first maps the received signal into a N dimensional vector. ($r(t) \rightarrow r$).
- The decision region is determined by the perpendicular bisectors of the signal points.
- Then the receiver finds which signal is closest (in Euclidean distance) to the received vector. (Find i for which $r \in R_i$).

Example



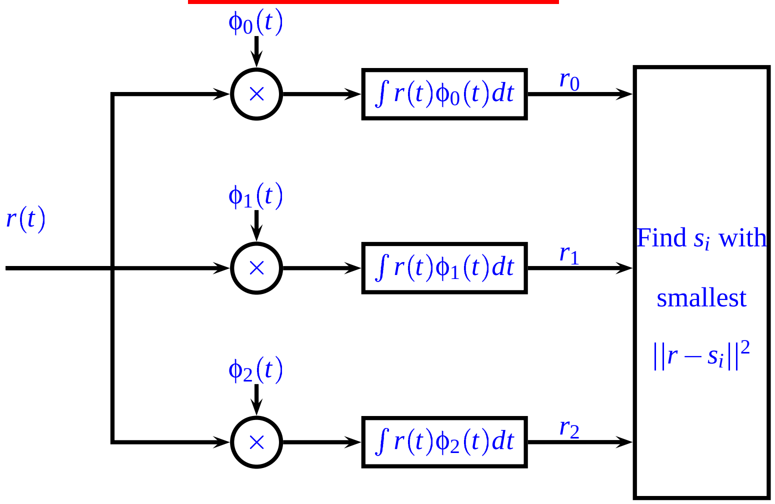
Orthonormal Basis Functions



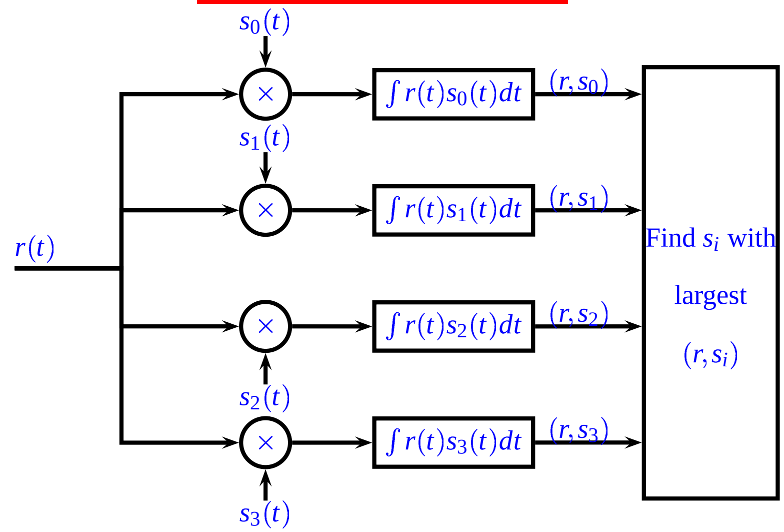
Signal Vectors

$$
\begin{aligned}
s_0 &= (+1,+1,+1) \\
s_1 &= (-1,+1,-1) \\
s_2 &= (+1,-1,-1) \\
s_3 &= (-1,-1,+1)
\end{aligned}
$$

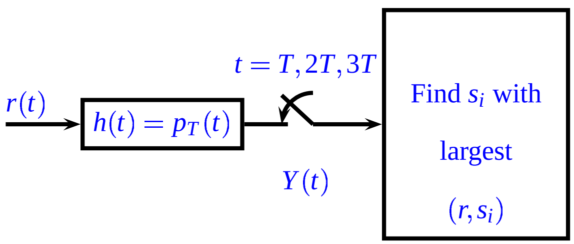
Optimum Receiver 1



Optimum Receiver 2



Optimum Receiver 3



$$
\begin{aligned}
r_0 = Y(T) &= \int r(t)\phi_0(t)dt = \int_0^T r(t)dt \\
r_1 = Y(2T) &= \int r(t)\phi_1(t)dt = \int_T^{2T} r(t)dt \\
r_2 = Y(3T) &= \int r(t)\phi_2(t)dt = \int_{2T}^{3T} r(t)dt
\end{aligned}
$$

Simplified Calculation

$$
\begin{aligned}
(r,s_0) &= +r_0+r_1+r_2 \\
(r,s_1) &= -r_0+r_1-r_2 \\
(r,s_2) &= +r_0-r_1-r_2 \\
(r,s_3) &= -r_0-r_1+r_2
\end{aligned}
$$

First calculated x_0,x_1,x_2,x_3 as follows

$$
\begin{aligned}
x_0 &= +r_0 \\
x_1 &= -r_0 \\
x_2 &= r_1+r_2 \\
x_3 &= r_1-r_2
\end{aligned}
$$

Then

$$
\begin{aligned}
(r, s_0) &= x_0 + x_2 \\
(r, s_1) &= x_1 + x_3 \\
(r, s_2) &= x_0 - x_2 \\
(r, s_3) &= x_1 - x_3
\end{aligned}
$$

Thus the calculation requires only 6 additions/subtractions.