Lecture 8

Goals

- Be able to analyze MPSK modualtion
- Be able to analyze QAM modualtion
- Be able to quantify the tradeoff between data rate and energy.

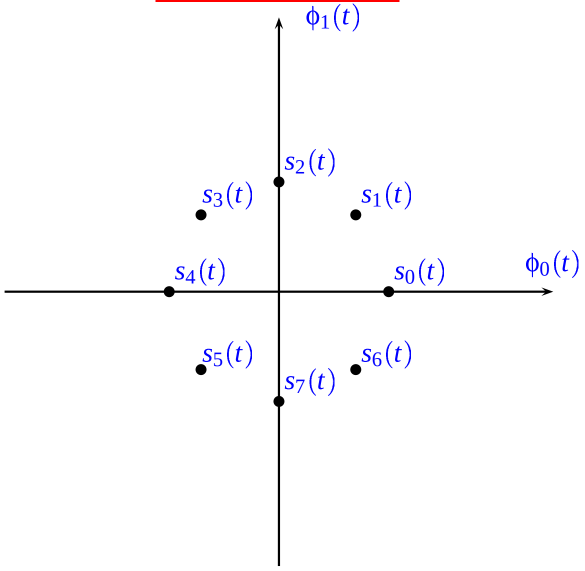
Multiphase Shift Keying (MPSK)

$$
\begin{aligned}
s_i(t) &= \sqrt{2P}\cos\left[2\pi f_c t + \frac{2\pi}{M}i\right] p_T(t) \quad 0 \le t \le T \\
&= A_{c,i}\sqrt{\frac{2}{T}}\cos(2\pi f_c t)p_T(t) - A_{s,i}\sqrt{\frac{2}{T}}\sin(2\pi f_c t)p_T(t) \\
&= A_{c,i}\phi_0(t) + A_{s,i}\phi_1(t)
\end{aligned}
$$

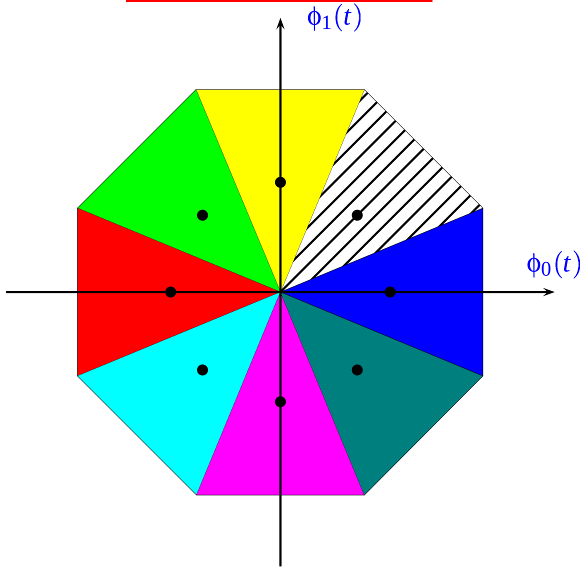
for $i = 0, 1, \ldots, M-1$, where $\phi_0(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)p_T(t)$ and $\phi_1(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_0 t)p_T(t)$.

$$
\begin{aligned}
A_{c,i} &= \sqrt{E}\cos(\frac{2\pi i}{M}) \\
A_{s,i} &= \sqrt{E}\sin(\frac{2\pi i}{M})
\end{aligned}
$$

Constellation



Decision Regions



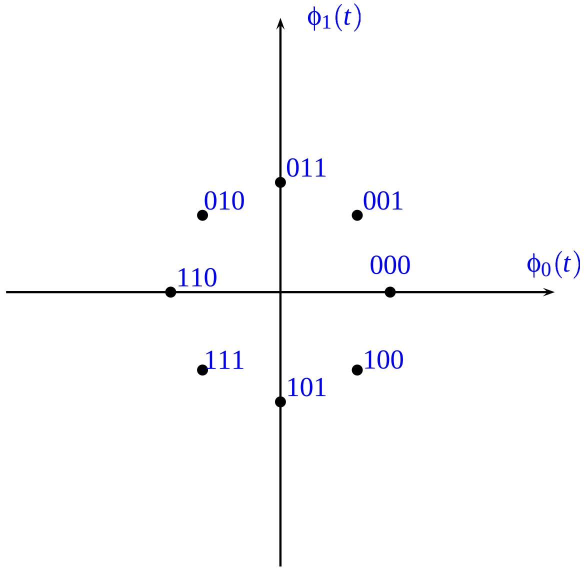
Notes on MPSK

For this modulation scheme we should use Gray coding to map bits into signals.

$$M=2 \Rightarrow \text{BPSK} \qquad M=4 \Rightarrow \text{QPSK}$$

This type of modulation has the properties that all signals have the same power thus the use of nonlinear amplifiers (class C amplifiers) affects each signal in the same manner. Furthermore if we are restricted to two dimensions and every signal must have the same power then this signal set minimizes the error probability of all such signal sets.

QPSK and BPSK are special cases of this modulation.



Symbol Error Probability for MPSK

The error probability for MPSK can be determined as follows. Consider a signal 0 transmitted where with the constellation shown above. The probability of error given signal 0 transmitted is the probability that the noise brings the received signal outside the region R_0 where the decision is that signal 0 was transmitted. This is given as

$$\begin{aligned}
P_{e,0} &= \int\int_{R_0^c} f_{Z_0|H_0}(z_0|H_0) f_{Z_1|H_0}(z_1|H_0) dz_0 dz_1 \\
&= \int\int_{R_0^c} \frac{1}{2\pi\sigma^2} \exp\{-\frac{1}{2\sigma^2}[(z_0-\sqrt{E})^2+z_1^2]\} dz_0 dz_1 \\
&= \int\int_{R_0^c+(\sqrt{E},0)} \frac{1}{2\pi\sigma^2} \exp\{-\frac{1}{2\sigma^2}[z_0^2+z_1^2]\} dz_0 dz_1 \\
&= 2\int_{\psi=0}^{\pi(1-\frac{1}{M})} \int_{r=R(\psi)}^{\infty} \frac{1}{2\pi} \frac{r}{2\sigma^2} \exp\{-\frac{r^2}{2\sigma^2}\} dr d\psi
\end{aligned}$$

$$\begin{aligned}
&= 2\int_{\psi=0}^{\pi(1-\frac{1}{M})} \frac{1}{2\pi} \int_{r=R(\psi)}^{\infty} \frac{r}{2\sigma^2} \exp\{-\frac{r^2}{2\sigma^2}\} dr d\psi \\
&= 2\int_{\psi=0}^{\pi(1-\frac{1}{M})} \frac{1}{2\pi} \int_{u=R^2(\psi)/2\sigma^2}^{\infty} \exp\{-u\} du d\psi \\
&= 2\int_{\psi=0}^{\pi(1-\frac{1}{M})} \frac{1}{2\pi} \exp\{-R^2/2\sigma^2\} d\psi
\end{aligned}$$

where R_0^c is the complement of R_0 and R is the distance from the signal point s_0 to the line with slope π/M.

$$\begin{aligned}
R &= \frac{\sqrt{E}\sin(\pi/M)}{\sin(\pi-(\psi+\pi/M))} \\
&= \frac{\sqrt{E}\sin(\pi/M)}{\sin(\psi+\pi/M)}
\end{aligned}$$

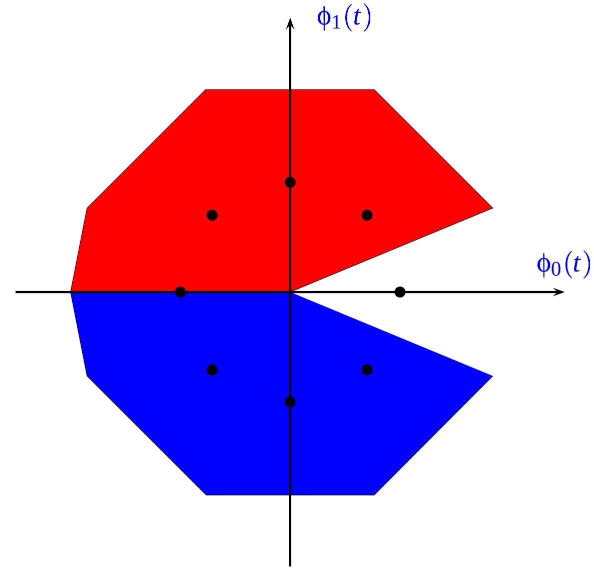
Thus

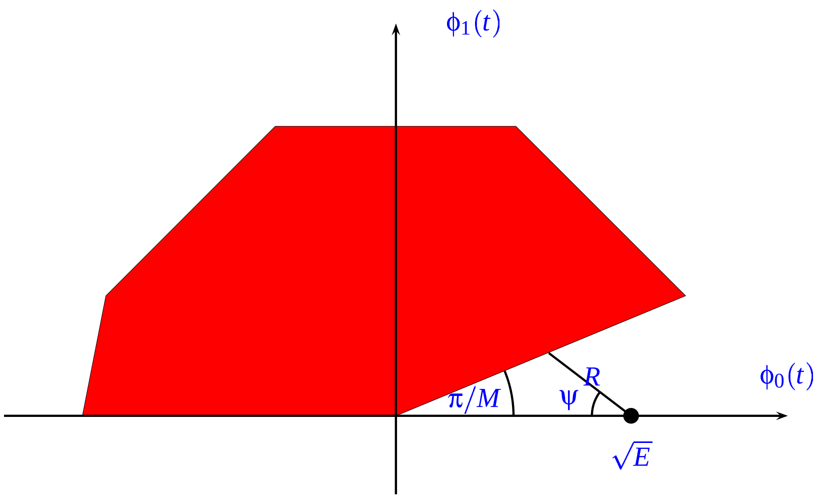
$$\frac{R^2}{2\sigma^2} = \frac{E\sin^2(\pi/M)}{N_0\sin^2(\psi+\pi/M)}$$

In the derivation of the error probability the last line follows from a change of variables where the point z_0, z_1 is mapped to an angle ϕ from the horizontal axis with reference point s_0 and a magnitude r from the point s_0. The symbol error probability is thus

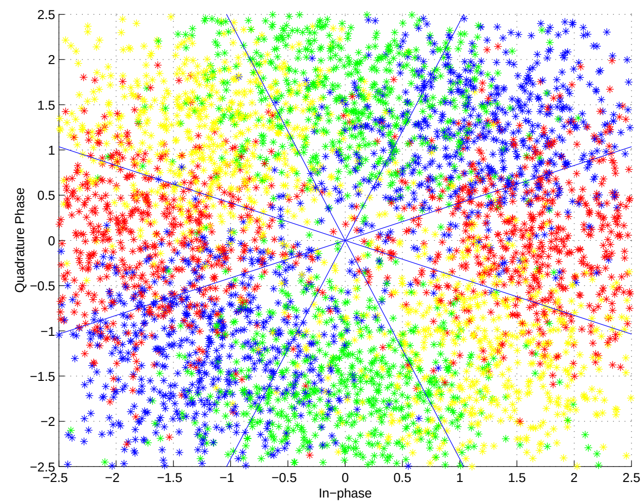
$$P_{e,s} \quad = \quad 2\int_{\psi=0}^{\pi(1-\frac{1}{M})} \frac{1}{2\pi}\exp\{-\frac{E\sin^2(\pi/M)}{N_0\sin^2(\psi+\pi/M)}\}d\psi$$

$$E_b = E/\log_2(M)$$

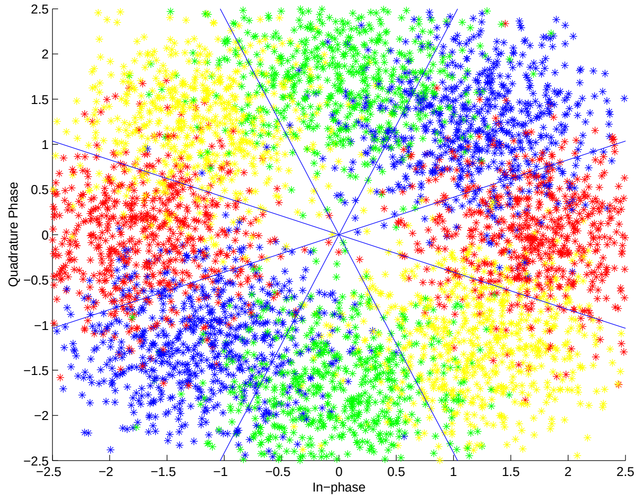




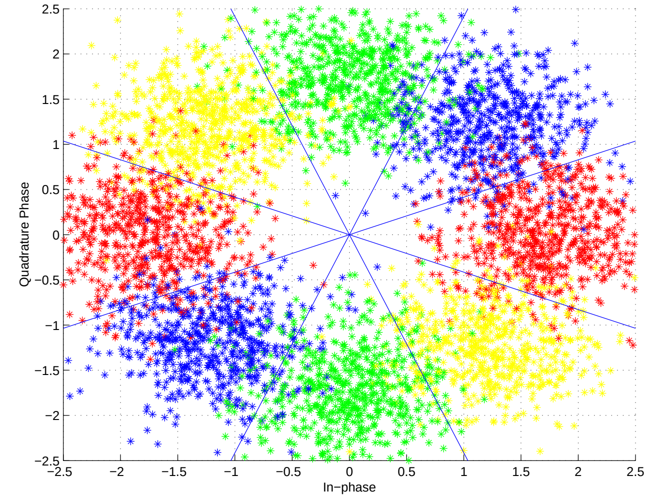
Constellation ($E_b/N_0 = 0$dB)



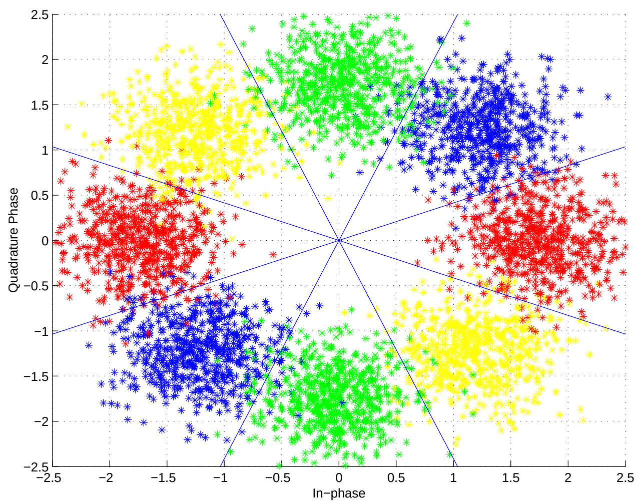
Constellation ($E_b/N_0 = 2$dB)



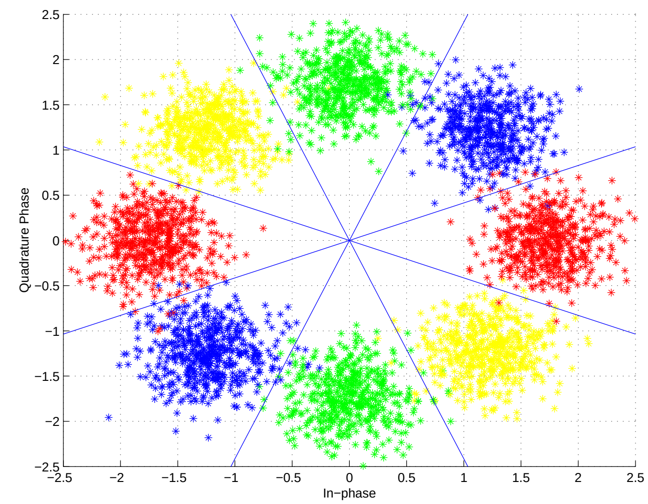
Constellation ($E_b/N_0 = 4$dB)



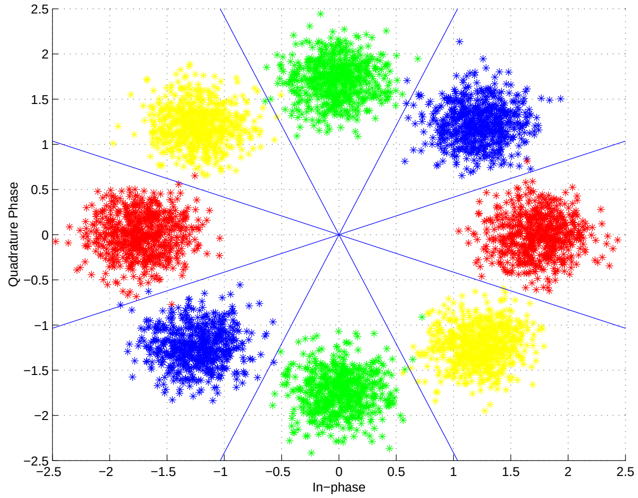
Constellation ($E_b/N_0 = 6$dB)



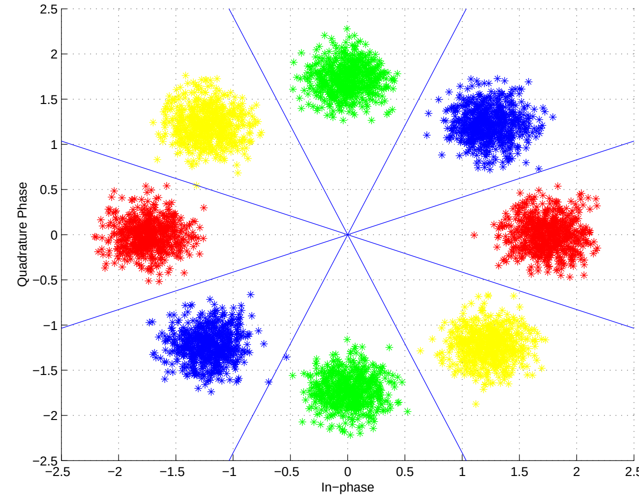
Constellation ($E_b/N_0 = 8$dB)



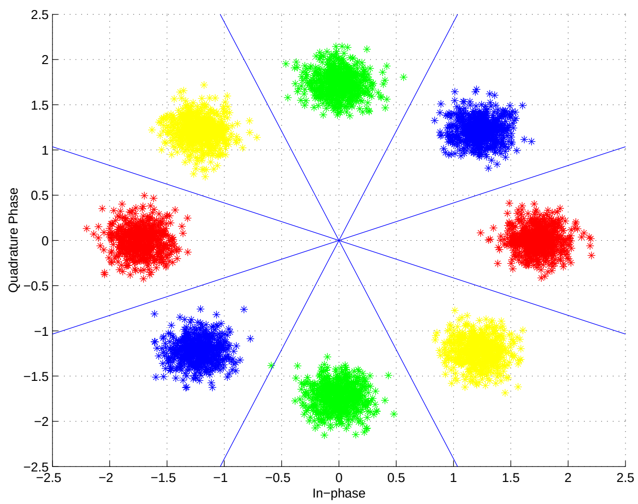
Constellation ($E_b/N_0 = 10$dB)



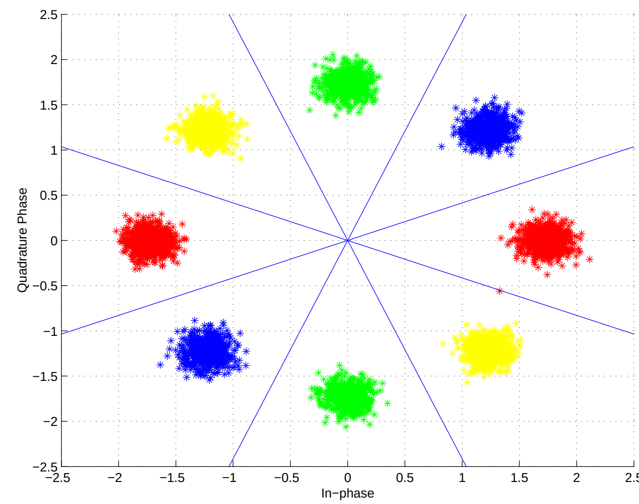
Constellation ($E_b/N_0 = 12$dB)



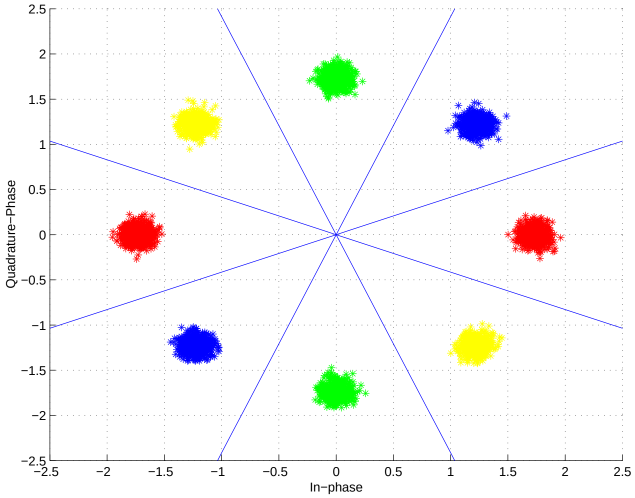
Constellation ($E_b/N_0 = 14$dB)



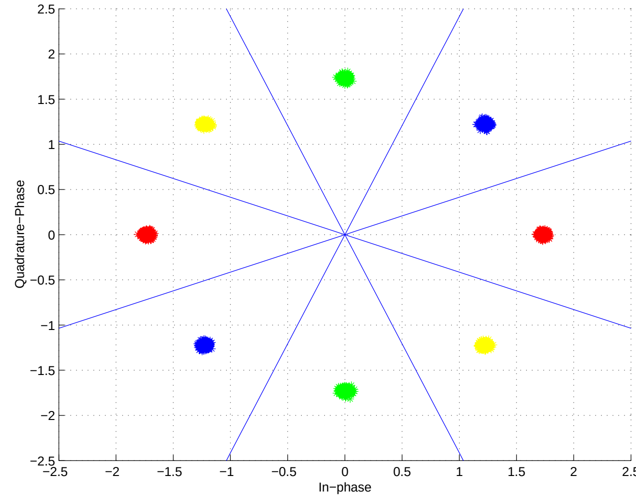
Constellation ($E_b/N_0 = 16$dB)



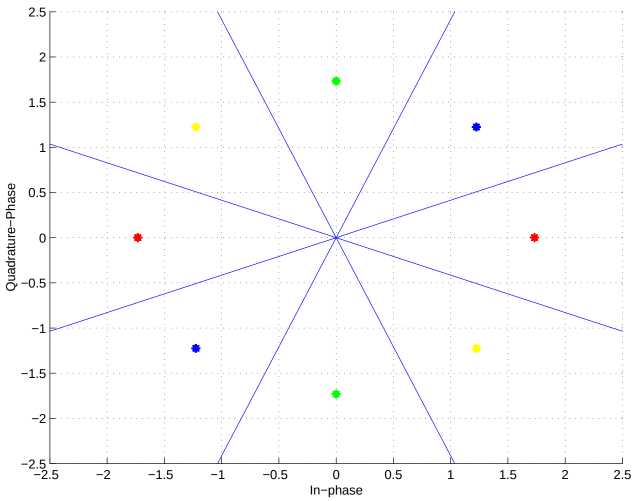
Constellation ($E_b/N_0 = 20$dB)



Constellation ($E_b/N_0 = 30$dB)



Constellation ($E_b/N_0 = 50$dB)



Symbol Error Probability

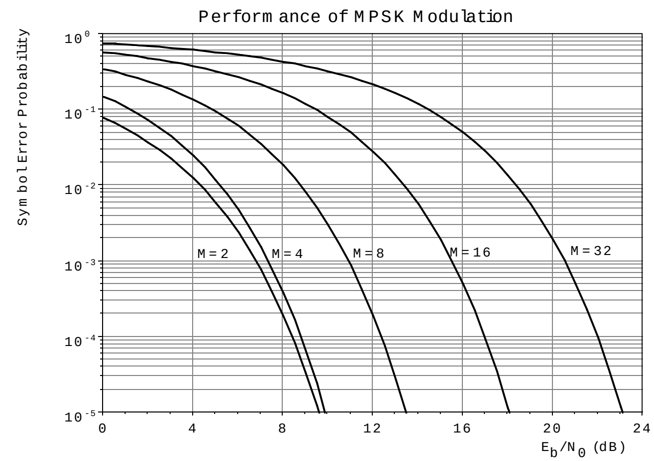


Figure 39: Symbol Error Probability for MPSK Signaling

Bit Error Probability

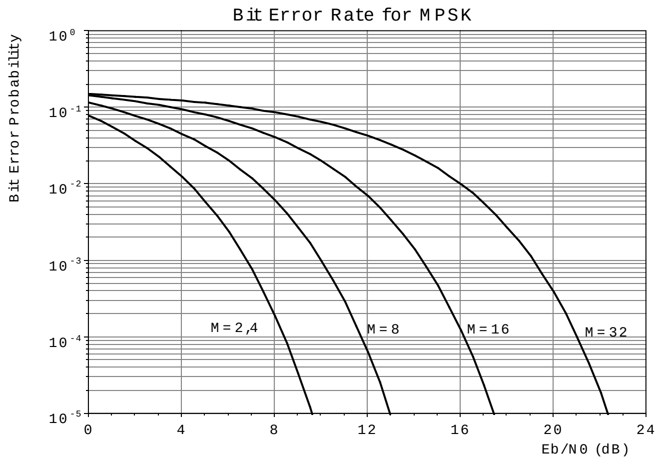


Figure 40: Bit Error Probability for MPSK Signaling (with Gray coding)

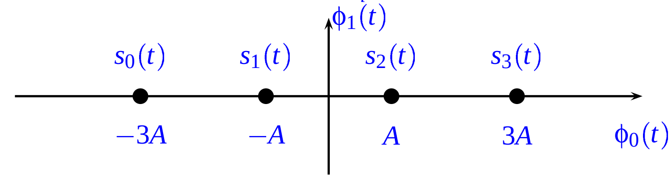
M-ary Pulse Amplitude Modulation (PAM)

$$s_i(t) = A_i\,\phi_0(t), \qquad 0 \le t \le T$$

where

$$A_i = (2i+1-M)A \qquad i = 0,1,\ldots,M-1$$

$$E_i = A_i^2$$



$$\overline{E} = \frac{1}{M}\sum_{i=0}^{M-1} E_i = \frac{A^2}{M}\sum_{i=0}^{M-1}(2i+1-M)^2 = \frac{M^2-1}{3}A^2$$

Error Probability

The error probability (for 4-ary) is

$$P_{e,1} = P_{e,2} = 2Q\left(\frac{A}{\sqrt{N_0/2}}\right)$$

$$P_{e,0} = P_{e,3} = Q\left(\frac{A}{\sqrt{N_0/2}}\right)$$

The average error probability (for 4-ary PAM) is

$$\begin{aligned}\bar{P_e} &= \frac{1}{4}P_{e,0} + \frac{1}{4}P_{e,1} + \frac{1}{4}P_{e,2} + \frac{1}{4}P_{e,3} \\ &= \frac{3}{2}Q\left(\frac{A}{\sqrt{N_0/2}}\right) = \frac{3}{2}Q\left(\sqrt{\frac{2\bar{E}}{5N_0}}\right)\end{aligned}$$

In general the error probability is

$$\begin{aligned}P_{e,s} &= \left(\frac{2(M-1)}{M}\right) Q\left(\sqrt{\frac{6\overline{E}}{(M^2-1)N_0}}\right) \\ &= \left(\frac{2(M-1)}{M}\right) Q\left(\sqrt{\frac{6\overline{E}_b\log_2(M)}{(M^2-1)N_0}}\right)\end{aligned}$$

Symbol Error Probability

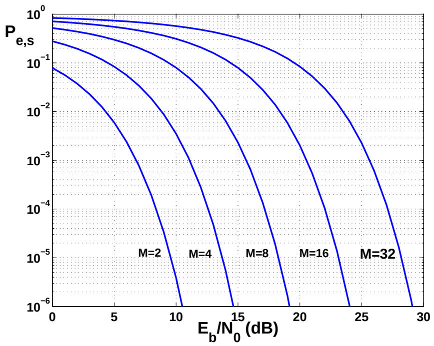


Figure 41: Symbol Error Probability for MPAM Signaling

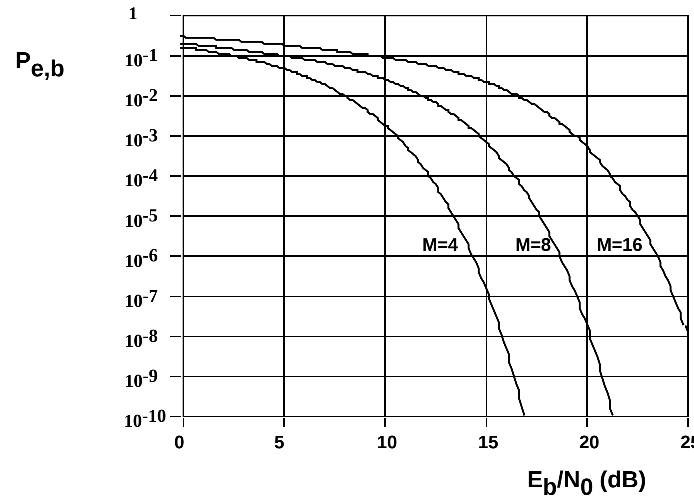


Figure 42: Bit Error Probability for MPAM Signaling

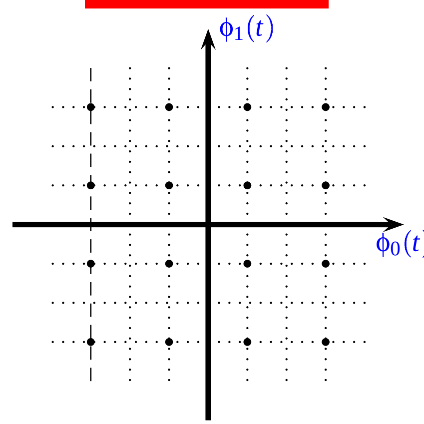
Quadrature Amplitude Modulation

For $i = 0, \ldots, M-1$

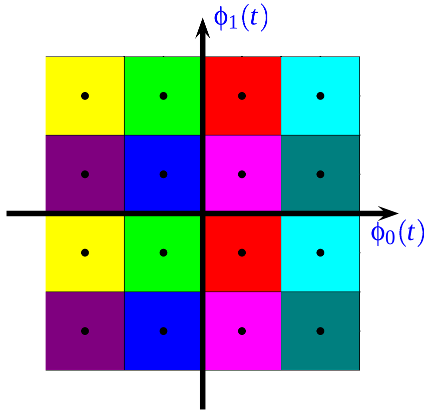
$$s_i(t) = A_i \cos 2\pi f_c t + B_i \sin 2\pi f_c t \qquad 0 \le t \le T$$

$$s_i(t) = A_i \sqrt{\frac{T}{2}} \phi_0(t) + B_i \sqrt{\frac{T}{2}} \phi_1(t) \qquad 0 \le t \le T$$

Constellation



Decision Regions



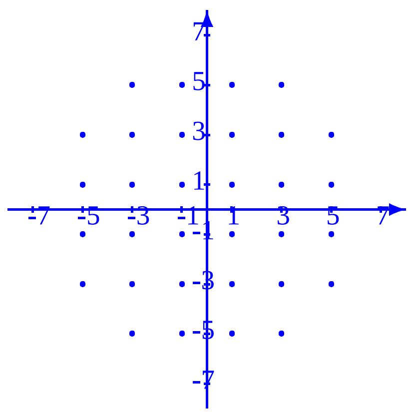
Error Probabilities for QAM

Since this is two PAM systems in quadrature. $P_{e,2} = 1-(1-P_{e,1})^2$ for PAM with $\sqrt{M}$ signals

The application of QAM is to bandwidth constrained channels. We can consider as a baseline a two dimensional modulation system transmitting a 2400 symbols per second. If each symbol represents 4 bits of information then the data rate is 9600 bits per second. So we would like to have more signals per dimension in order to increase the data rate. However, we must try to keep the signals as far apart from each other as possible (in oder to keep the error rate low). So an increase of the size of the signal constellation for fixed minimum distance would likely increase the total signal energy transmitted.

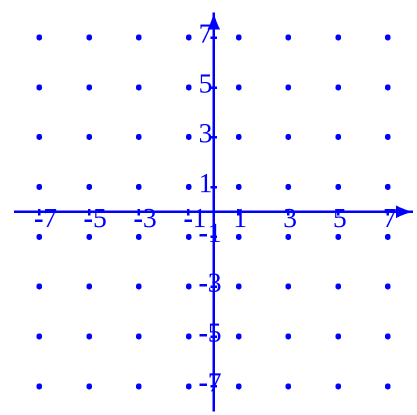
32-ary signal sets

Consider a 32-ary QAM signal set shown below. The average energy is 20. The minimum distance is 2 and the rate is 5 bits/dimension.

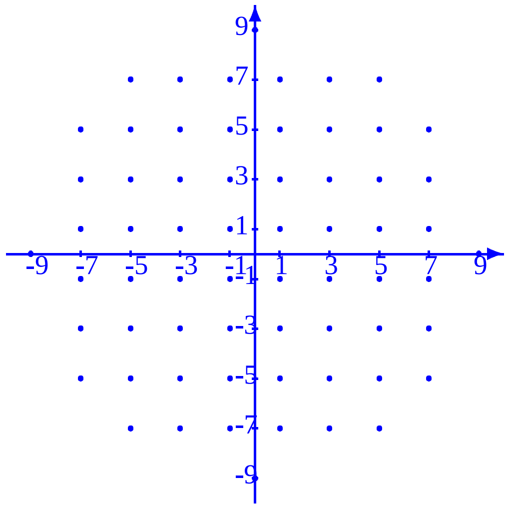


64-ary signal sets

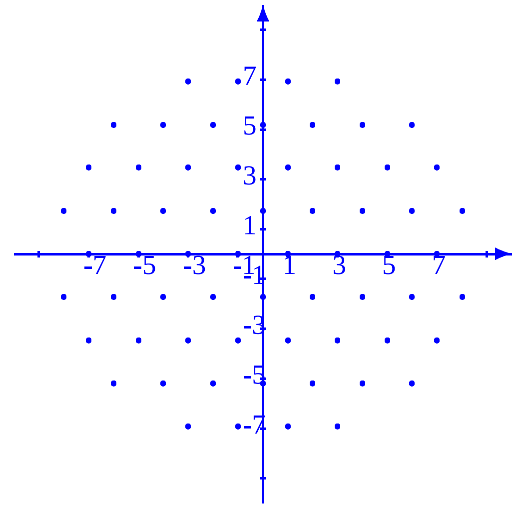
Consider a 64-ary QAM signal set shown below. The average energy is 42.



Modified QAM (used in Paradyne 14.4kbit modem). This has average energy of 40.9375.



The following hexagonal constellation has energy 35.25 but each interior point now has 6 neighbors compared to the four neighbors for the rectangular structures.



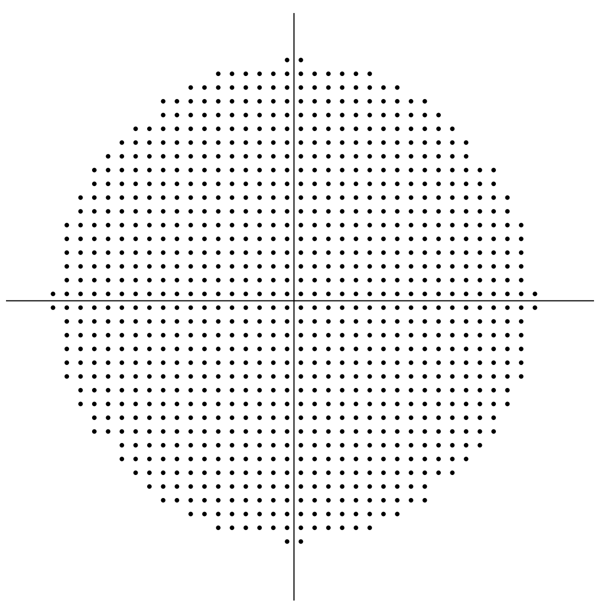
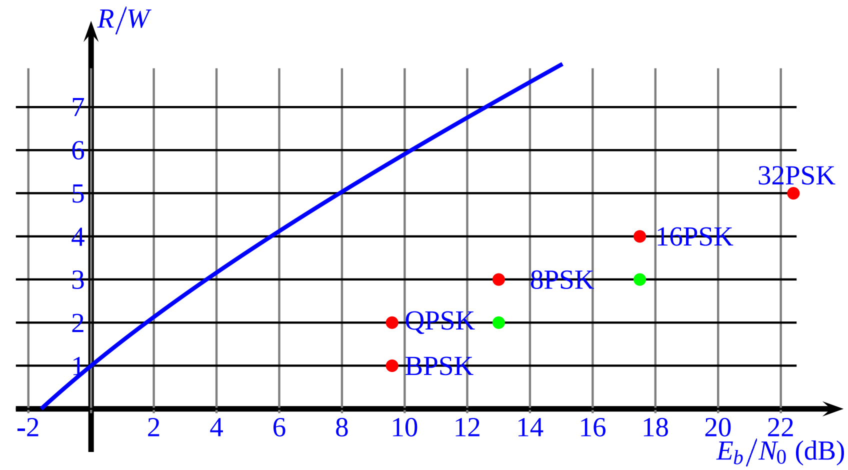
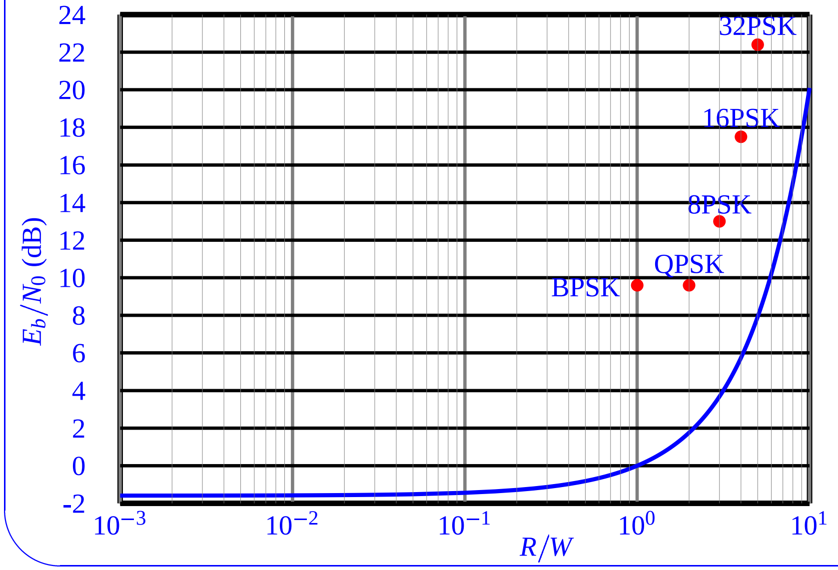


Figure 43: Constellation used in v34 28.8kbps modems (960 points)

Capacity



Capacity



32PSK
16PSK
8PSK
QPSK
BPSK
24
22
20
18
16
14
12
10
8
6
4
2
0
-2
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
R/W
E_b/N_0 (dB)