Lecture 9

Goals

- Be able to determine bandwidth efficiency and energy efficiency of orthogonal signals.
- Be able to demodulate orthogonal signals without phase reference.
- Be able to synthesize different types of orthogonal signals.

Orthogonal Signals

A set of signals $\{\phi_i(t) : 0 \le t \le T, 0 \le i \le M-1\}$ are said to be orthogonal (over the interval $[0,T]$) if

$$\int_0^T \phi_i(t)\phi_j(t)dt = 0, \qquad i \neq j.$$

In most cases the signals will have the same energy and it is convenient to normalize the signals to unit energy. A set of signals $\{\phi_i(t) : 0 \le t \le T, 1 \le i \le M\}$ are said to be orthonormal (over the interval $[0,T]$) if

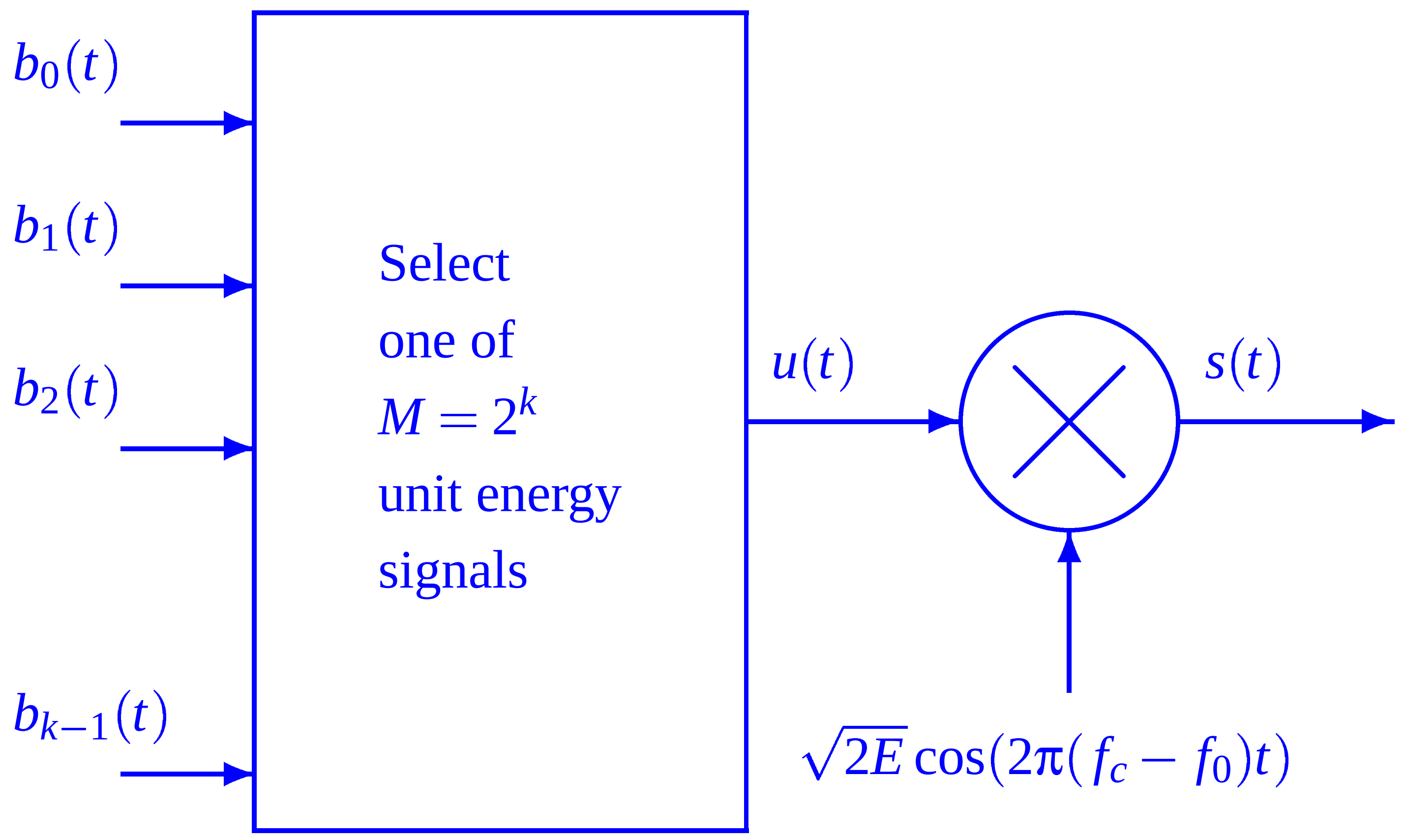
$$\int_0^T \phi_i(t)\phi_j(t)dt = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

The set of orthogonal signals can be described by

$$
\begin{aligned}
s_0(t) &= \sqrt{E}\phi_0(t) \\
s_1(t) &= \sqrt{E}\phi_1(t) \\
s_2(t) &= \sqrt{E}\phi_2(t) \\
s_{M-1}(t) &= \sqrt{E}\phi_{M-1}(t)
\end{aligned}
$$

Below we describe a number of different orthonormal signal sets. The signal sets will all be described at some intermediate frequency f_0 but are typically modulated up to the carrier frequency f_c.

General Modulator



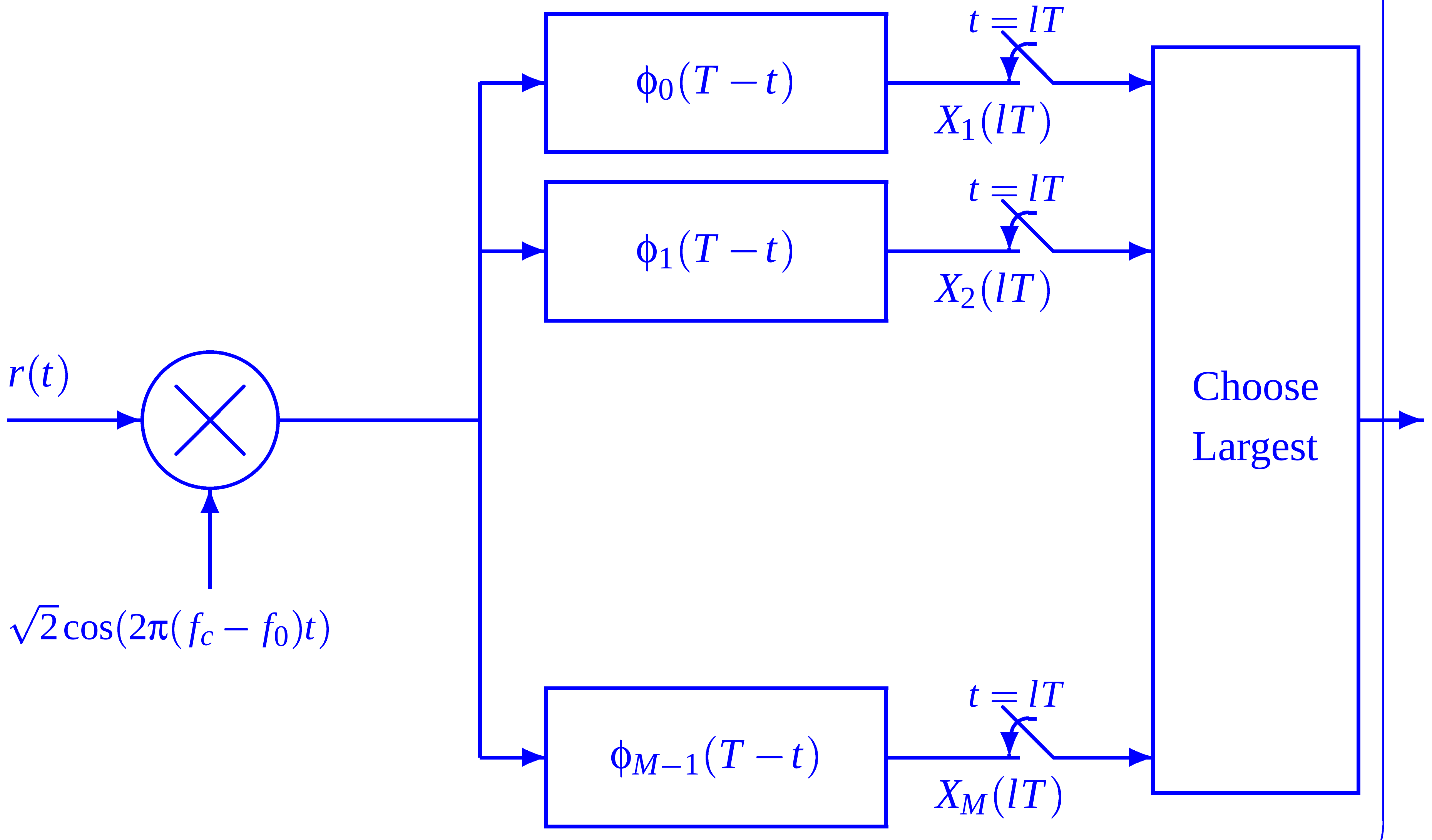
$$b_i(t) = \sum_{l=-\infty}^{\infty} b_l p_T(t - lT), i = 1, 2, \ldots, k$$

$$u(t) = \sum_{l=-\infty}^{\infty} \phi_{i_l}(t - lT)$$

where for $(l-1)T \leq t < T$

$$i_l = \begin{cases} 1, & b_1(t) = b_2(t) = \ldots = b_{k-1}(t) = b_k(t) = +1 \\ 2, & b_1(t) = b_2(t) = \ldots = b_{k-1}(t) = +1, b_k(t) = -1 \\ \\ M, & b_1(t) = b_2(t) = \ldots = b_{k-1}(t) = b_k(t) = -1 \end{cases}$$

General Coherent Demodulator



$\phi_m(T-t)$ is the impulse response of the m-th matched filter. The output of these filters (assuming that the i_l-th orthogonal signal is transmitted is) given by

$$X_m(lT) = \begin{cases} \eta_m, & m \neq i_l \\ \sqrt{E} + \eta_m, & m = i_l \end{cases}$$

where $\{\eta_m, m = 0, 1, 2, ..., M-1\}$ is a sequence of independent, identically distributed Gaussian random variables with mean zero and variance $N_0/2$.

Error Probability

To determine the probability of error we need to determine the probability that the filter output corresponding to the signal present is smaller than one of the other filter outputs.

$$
\begin{aligned}
P_{e,0} &= P\{X_0 < \max(X_1,\ldots,X_{M-1}) | s_0 \text{trans}\} \\
P_{e,0} &= 1 - P\{X_1 < X_0, X_2 < X_0, X_3 < X_0, \ldots, X_{M-1} < X_0 | s_0 \text{trans}\} \\
&= 1 - P\{\eta_1 < \sqrt{E} + \eta_0, \eta_2 < \sqrt{E} + \eta_0 + \ldots \eta_{M-1} < \sqrt{E} + \eta_0\} \\
&= 1 - \int_{-\infty}^{\infty} \Phi^{M-1}(\sqrt{E} + x) f_{\eta_0}(x) du \\
&= 1 - \int_{-\infty}^{\infty} \Phi^{M-1}(\sqrt{E} + x) \frac{1}{\sqrt{\pi N_0}} \exp\{-\frac{x^2}{N_0}\} dx \\
&= \frac{M-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(u - \sqrt{\frac{2E}{N_0}}) \Phi^{M-2}(u) e^{-u^2/2} du
\end{aligned}
$$

where $\Phi(u)$ is the distribution function of a zero mean, variance 1, Gaussian random variable given by

$$\Phi(u) = \frac{1}{2\pi} \int_{-\infty}^{u} e^{-x^2/2} dx.$$

The last step in the derivation is obtained by using the integration by parts formula. The symbol error probability can be upper bounded as

$$P_{e,s} \leq \begin{cases} 1, & \frac{E}{N_0} \leq \ln M \\ \exp\left\{-\left(\sqrt{\frac{E}{N_0}} - \sqrt{\ln M}\right)^2\right\}, & \ln M \leq \frac{E}{N_0} \leq 4 \ln M \\ \exp\{-(\frac{E}{2N_0} - \ln M)\}, & \frac{E}{N_0} \geq 4 \ln M. \end{cases}$$

Normally a communication engineer is more concerned with the energy transmitted per bit rather than the energy transmitted per signal, E. If we let E_b be the energy transmitted per bit then these are related as follows

$$E_b = \frac{E}{\log_2 M}.$$

Thus the bound on the symbol error probability can be expressed in terms of the energy transmitted per bit as

$$P_{e,s} \leq \begin{cases} 1, & \frac{E_b}{N_0} \leq \ln 2 \\ \exp_2\left\{-\log_2 M\left(\sqrt{\frac{E_b}{N_0}} - \sqrt{\ln 2}\right)^2\right\}, & \ln 2 \leq \frac{E_b}{N_0} \leq 4\ln 2 \\ \exp_2\left\{-\log_2 M\left(\frac{E_b}{2N_0} - \ln 2\right)\right\}, & \frac{E_b}{N_0} \geq 4\ln 2 \end{cases}$$

where $\exp_2\{x\}$ denotes 2^x. Note that as $M \to \infty$, $P_e \to 0$ if $\frac{E_b}{N_0} > \ln 2 =$ -1.59dB.

So far we have examined the symbol error probability for orthogonal signals. Usually the number of such signals is a power of 2, e.g. 4, 8, 16, 32, ... If so then each transmission of a signal is carrying $k = \log_2 M$ bits of information. In this case a communication engineer is usually interested in the bit error probability as opposed to the symbol error probability. Assume signal 0 is transmitted corresponding to the data bits being (000...00). If an error occurs and the demodulator chooses one of the incorrect signals than each of the incorrect signals has the same probability. Thus the signal corresponding to data bits being (000...01) has the same probability as an error to a signal corresponding to data bits (111...11). If signal 0 is transmitted then there will be M/2 other signals that will cause a bit error in any particular bit. Thus

$$P_{e,b} = \frac{M}{2(M-1)} P_{e,s} = \frac{2^{k-1}}{2^k - 1} P_{e,s}.$$

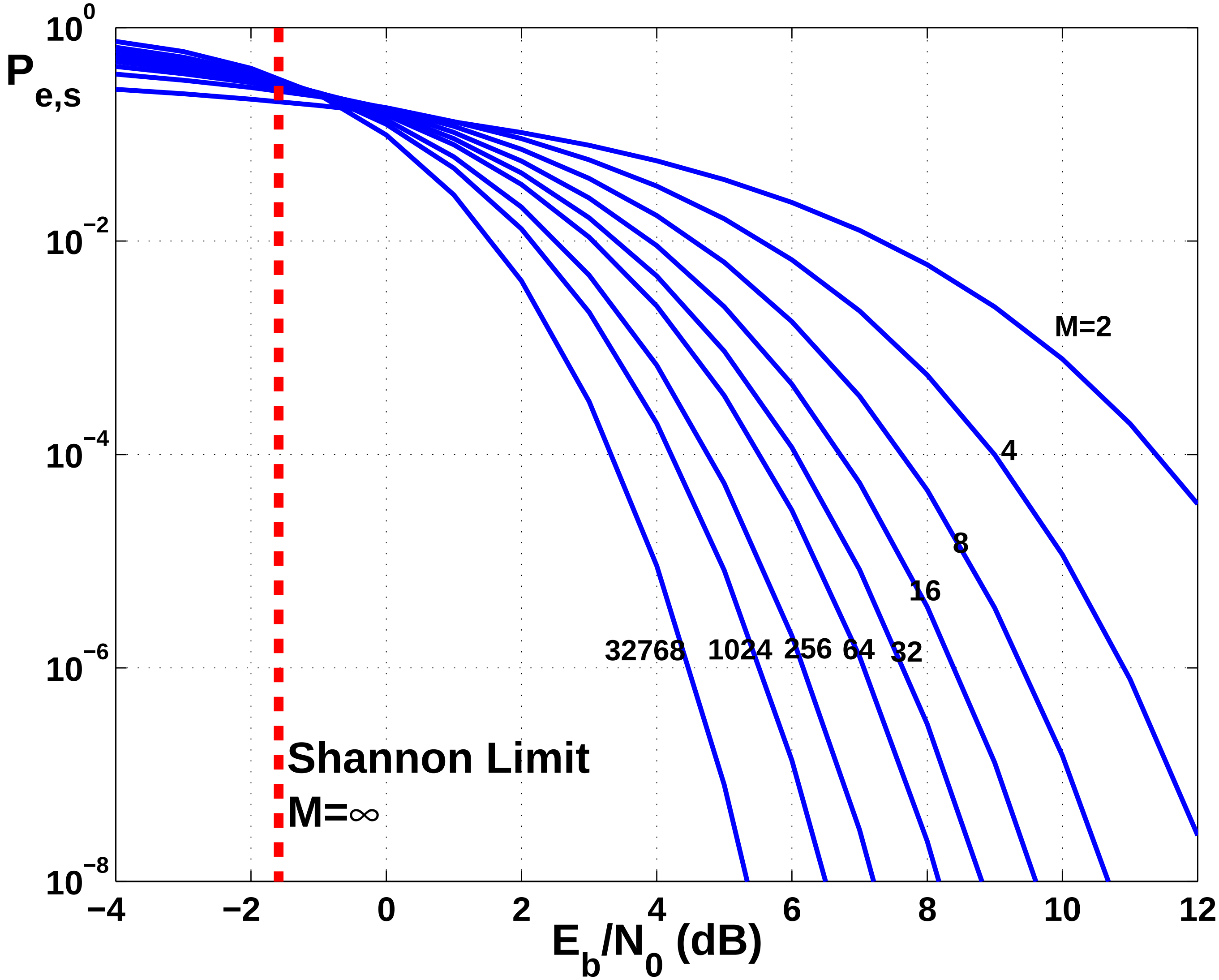


Figure 44: Symbol Error Probability for Coherent Demodulation of Orthogonal Signals

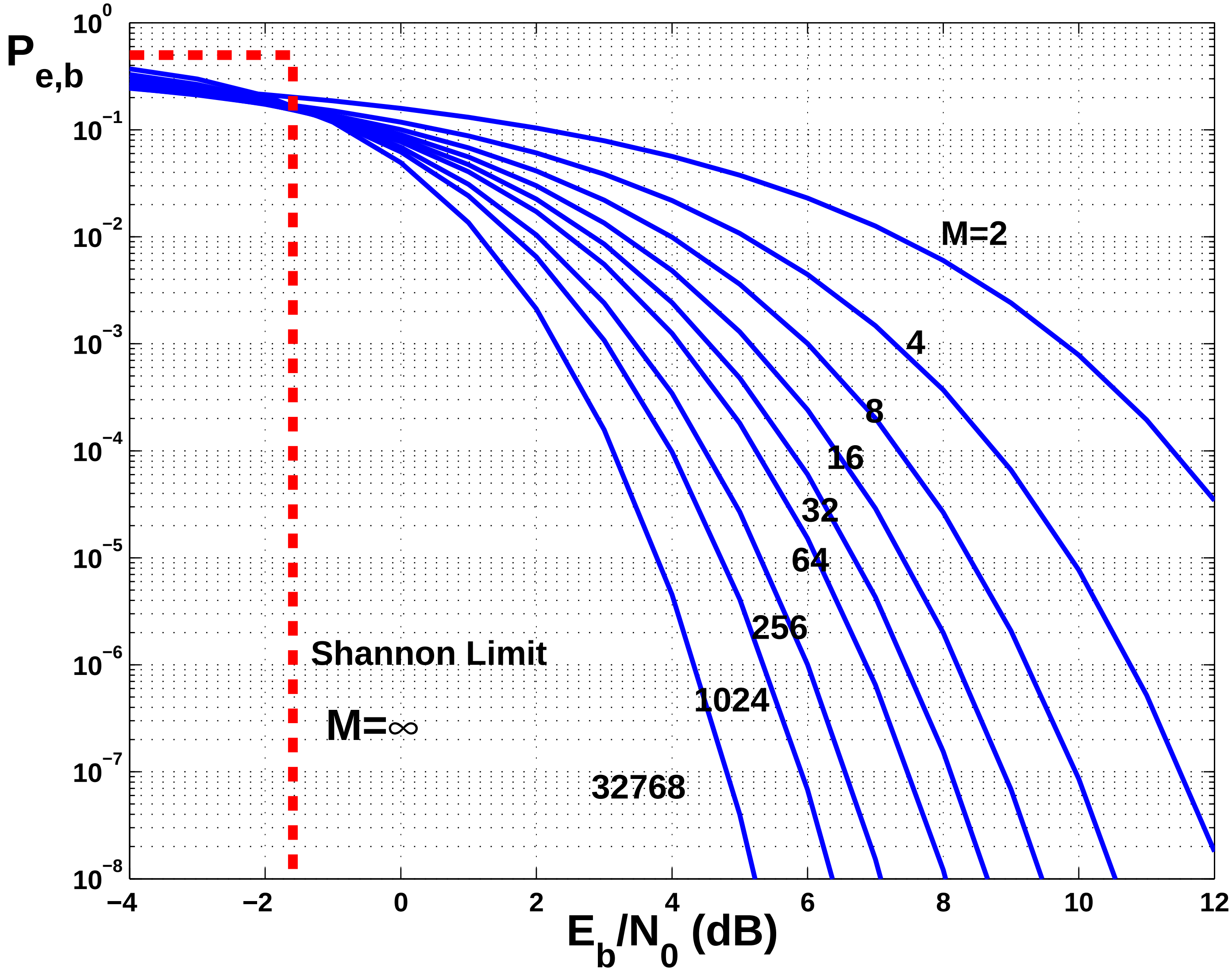


Figure 45: Bit Error Probability for Coherent Demodulation of Orthogonal Signals

Orthogonal Signal Sets

Below we define several different orthogonal signal sets. We will define the bandwidth of a signal set as the minimum difference in carrier frequencies between two such signal sets so that any signal from one set is orthogonal to any signal from the other set.

A. Time-orthogonal

$$\phi_i(t) = \begin{cases} \sqrt{\frac{2M}{T}}\sin(2\pi f_0 t), & \frac{iT}{M} \le t < (i+1)T/M \\ 0, & \text{elsewhere} \end{cases}$$

$$i = 0, 1, \ldots, M, \quad f_0 = n\frac{M}{2T}$$

$$\begin{aligned} W &= [\frac{(n+1)M}{2T} - \frac{nM}{2T}] \\ &= \frac{M}{2T} \end{aligned}$$

B. Time-orthogonal quadrature-phase

$$\phi_{2i}(t) = \begin{cases} \sqrt{\frac{2M}{T}}\sin(2\pi f_0 t), & \frac{2iT}{M} \leq t < 2(i+1)T/M \\ 0, & \text{elsewhere} \end{cases}$$

$$\phi_{2i+1}(t) = \begin{cases} \sqrt{\frac{2M}{T}}\cos(2\pi f_0 t), & \frac{2iT}{M} \leq t < 2(i+1)\frac{T}{M} \\ 0 & \text{elsewhere} \end{cases}$$

$$i = 0, 1, \ldots, \frac{M}{2} - 1, \quad M \text{ even}, \quad f_0 = n\frac{M}{2T}$$

$$\begin{aligned} W &= [\frac{(n+1)M}{2T} - \frac{nM}{2T}] \\ &= \frac{M}{2T} \end{aligned}$$

C. Frequency-orthogonal

$$
\begin{aligned}
\phi_i(t) &= \sqrt{\frac{2E}{T}}\sin[2\pi(f_0+\frac{i}{2T}t)], \qquad\qquad 0 \le t \le T \\
i &= 0,1,\ldots,M-1, \;\; f_0=\frac{nM}{2T}.
\end{aligned}
$$

$$
\begin{aligned}
W &= [\frac{(n+1)M}{2T}-\frac{nM}{2T}] \\
&= \frac{M}{2T}
\end{aligned}
$$

D. Frequency-orthogonal quadrature-phase

$$
\begin{aligned}
\phi_{2i}(t) &= \sqrt{\frac{2E}{T}}\sin[2\pi(f_0+\frac{i}{T}),t] \qquad && 0 \le t < T \\
\phi_{2i+1}(t) &= \sqrt{\frac{2E}{T}}\cos[2\pi(f_0+\frac{i}{T})t], \qquad && 0 \le t \le T
\end{aligned}
$$

$$f_0 = \frac{nM}{2T}.$$

$$
\begin{aligned}
W &= [\frac{(n+1)M}{2T} - \frac{nM}{2T}] \\
&= \frac{M}{2T}
\end{aligned}
$$

E. Hadamard-Walsh Construction

The last construction of orthogonal signals is done via the Hadamard Matrix. The Hadamard matrix is an N by N matrix with components either +1 or -1 such that every pair of distinct rows are orthogonal. We show how to construct a Hadamard when the number of signals is a power of 2 (which is often the case).

Begin with a two by two matrix

$$H_2 = \begin{bmatrix} +1 & +1 \\ +1 & -1 \end{bmatrix}.$$

Then use the recursion

$$H_{2^l} = \begin{bmatrix} +H_{2^{(l-1)}} & +H_{2^{(l-1)}} \\ +H_{2^{(l-1)}} & -H_{2^{(l-1)}} \end{bmatrix}.$$

Now it is easy to check that distinct rows in these matrices are orthogonal. The i-th modulated signal is then obtained by using a single (arbitrary) waveform N times in nonoverlapping time intervals and multiplying by the $j-th$ repetition of the waveform by the jth component of the i-th row of the matrix.

Example ($M=4$):

$$
\begin{aligned}
H_4 &= \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} \\
&= \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & +1 \end{bmatrix}.
\end{aligned}
$$

Example ($M = 8$):

$$
\begin{aligned}
H_8 &= \begin{bmatrix} H_4 & H_4 \\ H_4 & -H_4 \end{bmatrix} \\
&= \begin{bmatrix} H_2 & H_2 & H_2 & H_2 \\ H_2 & -H_2 & H_2 & -H_2 \\ H_2 & H_2 & -H_2 & -H_2 \\ H_2 & -H_2 & -H_2 & H_2 \end{bmatrix}
\end{aligned}
$$

$$
= \begin{bmatrix}
+1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\
+1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 \\
+1 & +1 & -1 & -1 & +1 & +1 & -1 & -1 \\
+1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 \\
+1 & +1 & +1 & +1 & -1 & -1 & -1 & -1 \\
+1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 \\
+1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 \\
+1 & -1 & -1 & +1 & -1 & +1 & +1 & -1
\end{bmatrix}.
$$

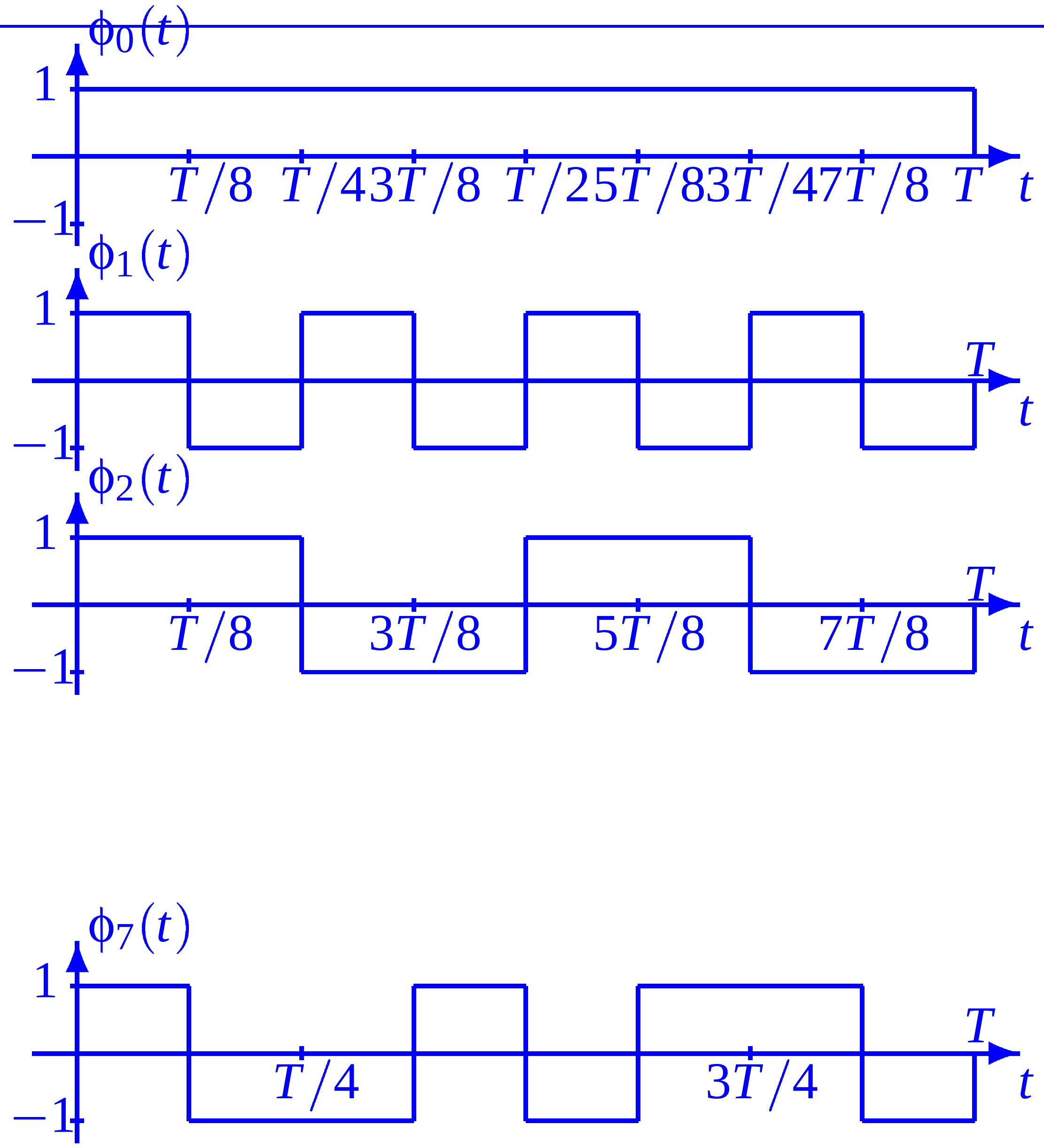


Figure 46: Hadamard-Walsh Orthogonal Signals

Processing of of Hadamard Generated Orthogonal Signals

$$
\begin{aligned}
W_0 &= (X_0 + X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7) \\
W_1 &= (X_0 - X_1 + X_2 - X_3 + X_4 - X_5 + X_6 - X_7) \\
W_2 &= (X_0 + X_1 - X_2 - X_3 + X_4 + X_5 - X_6 - X_7) \\
W_3 &= (X_0 - X_1 - X_2 + X_3 + X_4 - X_5 - X_6 + X_7) \\
W_4 &= (X_0 + X_1 + X_2 + X_3 - X_4 - X_5 - X_6 - X_7) \\
W_5 &= (X_0 - X_1 + X_2 - X_3 - X_4 + X_5 - X_6 + X_7) \\
W_6 &= (X_0 + X_1 - X_2 - X_3 - X_4 - X_5 + X_6 + X_7) \\
W_7 &= (X_0 - X_1 - X_2 + X_3 - X_4 + X_5 + X_6 - X_7)
\end{aligned}
$$

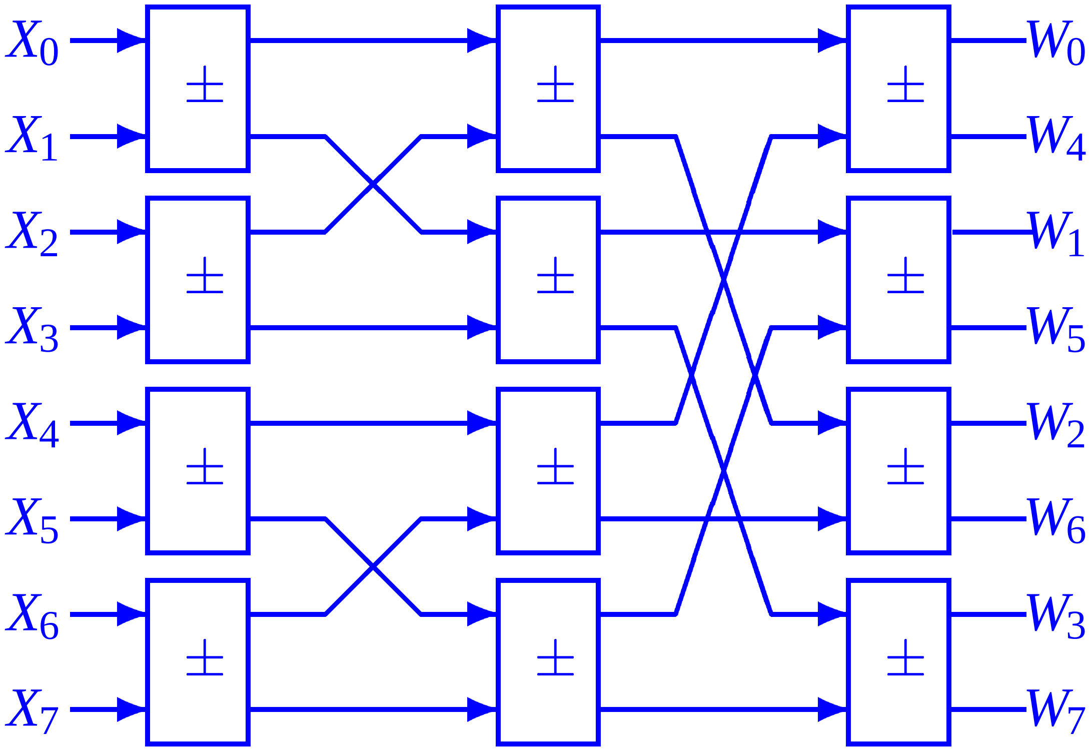


Figure 47: Fast Processing for Hadamard Signals

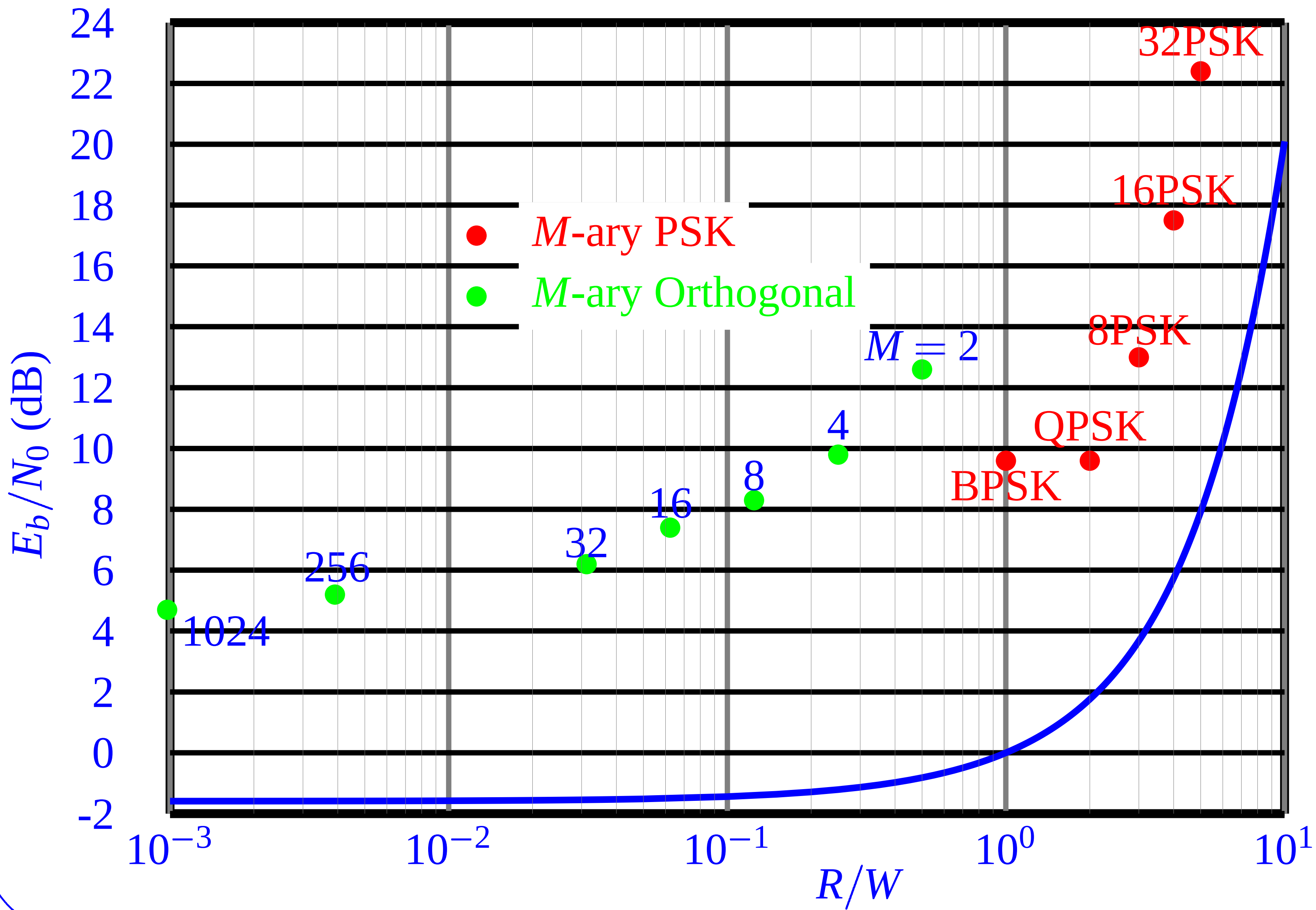
Bandwidth of Orthogonal Signals

If we define bandwidth of M signals as minimum frequency separation between two such signal sets such that any signal from one signal set is orthogonal to every signal from a frequency adjacent signal set then for all of these examples of M signals the bandwidth is

$$W = \frac{M}{2T} \Rightarrow M = 2WT!$$

Thus there are $2WT$ orthogonal signals in bandwidth W and time duration T.

Capacity



24
22
20
18
16
14
12
10
8
6
4
2
0
-2
E_b/N_0 (dB)
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
R/W
M-ary PSK
M-ary Orthogonal
32PSK
16PSK
8PSK
QPSK
BPSK
$M = 2$
4
8
16
32
256
1024