EECS 461, Fall 2009, Problem Set 4[1]

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1. We have seen that important properties of second order systems are described by the roots of the characteristic equation. If these roots are complex, it is useful to parameterize the location of these roots in the complex plane in terms of natural frequency and damping coefficient.

   The time response of a second order system to a unit step input (a constant input with magnitude equal to one) for various values of damping coefficient is plotted in Figure 1. Two measures are frequently used to describe this time response. One is the *rise time*, t_r, defined by the time it takes the system to travel

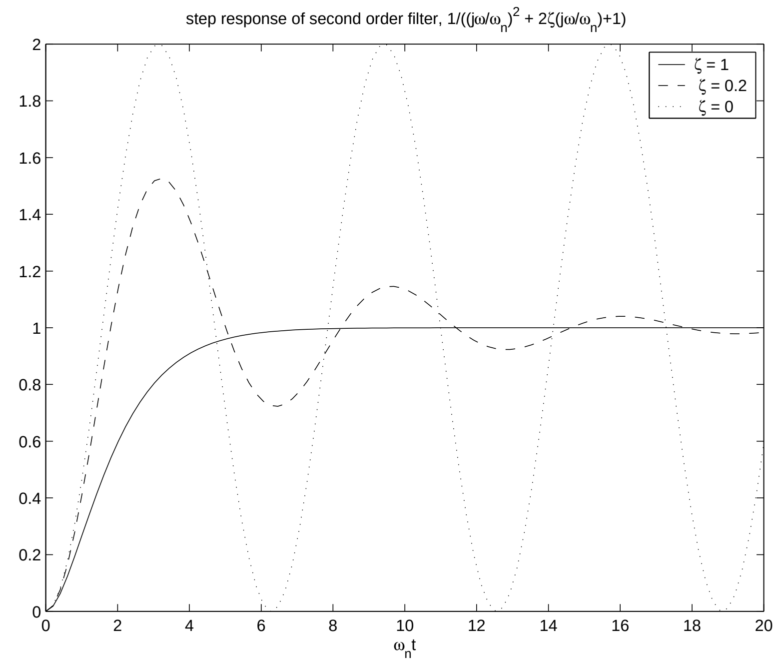

Figure 1: Step Response as a Function of Natural Frequency and Damping

   from 10% to 90% of its steady state value. It is possible to approximate the rise time by

$$t_r \approx \frac{1.8}{\omega_n}.$$

   Although it is possible to obtain a more accurate approximation by incorporating the effect of damping on t_r, the trend is clear: rise time is inversely proportional to ω_n, and thus to the bandwidth of the frequency response of the second order system.

   Another important measure of system response is the *overshoot*, defined by the amount that the peak in the step response exceeds its steady state value. Note from the figure that $\zeta = 1$ implies there is no overshoot and $\zeta = 0.2$ implies an overshoot of about 50%. Overshoot is undefined for $\zeta = 0$ because the system never reaches steady state. In general, the relation between damping and overshoot for a second order system is shown in Figure 2. Many systems of interest have more than two integrators, and are thus of order greater than two. In many cases, these systems can be approximated by a second order system. One example is the DC motor we have been studying. Consider the speed control feedback system for the DC motor, depicted in Figure 3. The controller in this case is an integral controller K/s. Recall that, with such a controller, the steady state response to a step change in the speed command has zero steady state error.

[1] Revised October 8, 2009.

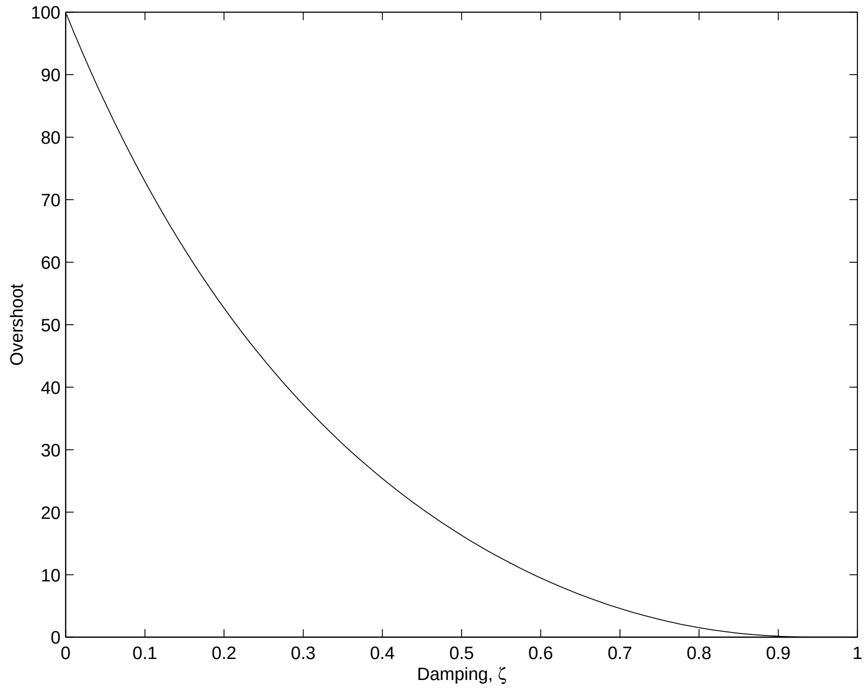

Figure 2: Overshoot vs. Damping

The characteristic equation has three roots: one of these corresponds to the circuit dynamics and is very fast. The dominant response is due to a complex pair of roots.

(a). Build a SIMULINK model as shown in Figure 3. Modify the m-file PS4_prob1.m to compute the natural frequency and damping for various values of K (including $K = 1, 10, 50, 100$). HAND IN: Your Simulink model, one set of four plots (Figure 4 from the Matlab file), and a table of natural frequency and damping of the complex roots.

(b). Use the discussion above to estimate the rise time and overshoot associated with these roots, and compare them to the values obtained from the time domain simulations. HAND IN: A table comparing the estimated values of rise time and overshoot to the actual values.

(c). How does the time response correlate with the closed loop frequency response? Describe briefly and qualitatively.

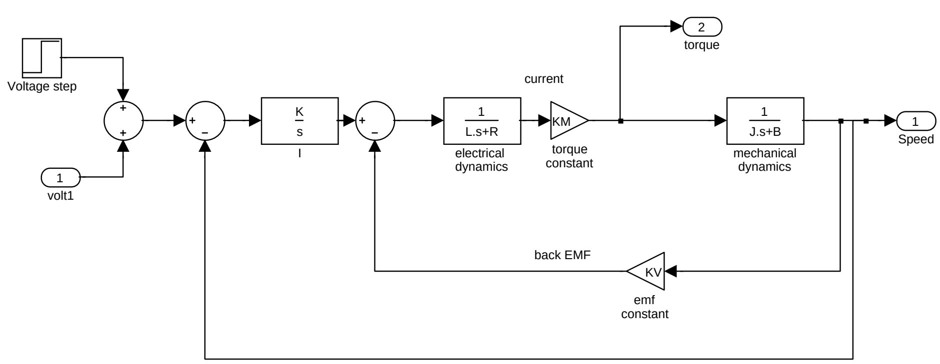

Figure 3: DC Motor with Speed Control

2. Consider the mass/spring/damper system shown in Figures 4-5. For simplicity, set $M = 1$.

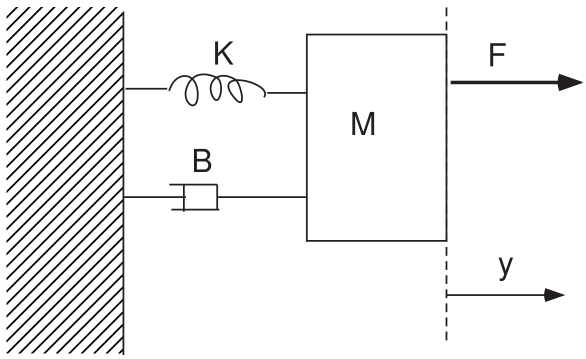

Figure 4: Spring Mass Damper System

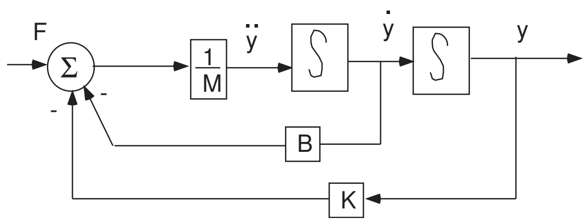

Figure 5: Block Diagram of Spring Mass Damper System

(a). Suppose that $B = 0$. What are the natural frequency and damping of the characteristic roots? How do these values vary with the spring constant K? Evaluate step response and frequency response plots for a few values of K. How do these plots change as K increases? HAND IN: One set of step and frequency response plots showing the dependence on K.

(b). Now fix $K = 2$, and suppose that $B \neq 0$. How do the natural frequency and damping vary with B? For what values of B are the roots real? imaginary? Again, evaluate step and frequency response plots for a few values of B and describe the trends you see. HAND IN: One set of step and frequency response plots showing the dependence on B.

You may use the Matlab file PS4_Prob2.m.

3. Consider the *unforced* response of a second order linear system to an initial condition:

$$a\ddot{x} + b\dot{x} + cx = 0, \quad x(0) = x_0,\ \dot{x}(0) = \dot{x}_0. \tag{1}$$

To derive the solution to this differential equation, we may use the techniques of Laplace transforms. Because the initial conditions are assumed to be nonzero, we must replace the derivatives of x as follows:

$$\dot{x} \Rightarrow sX(s) - x_0 \tag{2}$$

$$\ddot{x} \Rightarrow s^2X(s) - sx_0 - \dot{x}_0 \tag{3}$$

Substituting (2) and (3) into (1) and solving for $X(s)$ yields

$$X(s) = \frac{(as+b)x_0 + \dot{x}_0}{as^2 + bs + c}. \tag{4}$$

A precise expression for the time response may be found by calculating the inverse Laplace transform of (4). One almost never does this. Instead, one computes the time response via simulation. However, it is often important to know how the *qualitative* behavior of the solution depends upon the coefficients b and c. Define the *characteristic equation* of the differential equation by

$$as^2 + bs + c = 0. \tag{5}$$

The *characteristic roots* are the zeros of the characteristic equation (5):

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{6}$$

There are three possibilities for the response, depending on the sign of the *discriminant* $b^2 - 4ac$:

$$x(t) = A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t}, \quad b^2 - 4ac > 0, \quad \text{(distinct real roots, } \lambda_1 \neq \lambda_2) \tag{7}$$

$$x(t) = A_1e^{\lambda t} + A_2te^{\lambda t}, \quad b^2 - 4ac = 0, \quad \text{(repeated real roots, } \lambda_{1,2} = \lambda) \tag{8}$$

$$x(t) = A_1e^{\alpha t}\cos(\beta t) + A_2e^{\alpha t}\sin(\beta t), \quad b^2 - 4ac < 0, \quad \text{(complex roots, } \lambda_{1,2} = \alpha \pm j\beta) \tag{9}$$

We say that the system is *asymptotically stable* if the response to any initial condition decays to zero with time, *unstable* if the response to some initial condition increases without bound, and *marginally stable* if the response to initial conditions neither decays to zero nor becomes unbounded. It follows from (7)-(9) that the system is asymptotically stable precisely when the real parts of the characteristic roots are negative; i.e., when both roots lie in the open left half plane (OLHP). Furthermore, it is easy to verify that the characteristic roots lie in the OLHP if and only if the coefficients a, b, and c all have the same sign. (The relation between coefficients and stability for differential equations of higher order is more complicated.)

Define the damping coefficient

$$\zeta \triangleq \frac{b/a}{2\sqrt{c/a}}. \tag{10}$$

Note that the characteristic roots are real and distinct, real and repeated, or complex depending on whether the damping coefficient (10) satisfies $\zeta^2 > 1$, $\zeta^2 = 1$, or $\zeta^2 < 1$, respectively. If the roots are complex, then they are asymptotically stable if $0 < \zeta < 1$, and marginally stable if $\zeta = 1$.

We have seen that the qualitative behavior of the system in response to a step input is determined by the natural frequency, $\omega_n = \sqrt{c/a}$, and the damping ζ. These parameters also govern the behavior of the response to initial conditions. In particular, if the roots are complex, then the system is underdamped, and the response will tend to oscillate as it settles to its final value. If $\zeta = 1$, then the system is critically damped and, if $\zeta > 1$, the system is overdamped. In either of these cases, the response will not oscillate and will approach its final value monotonically.

Consider the virtual wall depicted in Figure 6. When the puck is to the left of the wall, it moves freely. When it is to the right of the wall, it moves as though it were attached to the wall with a spring and damping mechanism. To express the dynamical relations for the puck when it is "inside" the virtual wall, it is convenient to define a variable $\Delta z \triangleq z - z_w$. Newton's laws say that

$$M\Delta\ddot{z} = -K\Delta z - B\Delta\dot{z} \tag{11}$$

and rearranging, yields the form of (1):

$$\Delta\ddot{z} + (B/M)\Delta\dot{z} + (K/M)\Delta z = 0 \tag{12}$$

(a). Evaluate the damping and natural frequency in terms of the mass M, spring constant K, and damping constants B.

(b). Suppose that $M = 2$ and $K = 10$. For what values of B will the characteristic roots be real and distinct? real and repeated? complex?

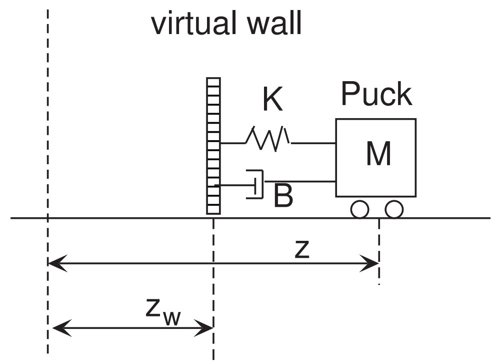

Figure 6: Virtual Wall with Spring and Damper

(c). Construct a SIMULINK model of the wall as shown in Figure 7. Set $K = 10$, and simulate the wall with no damping ($B = 0$), with underdamping, and with overdamping. For what values of B will the puck never leave the wall? Why? Explain. Use the m-file "prob3_PS4.m". Notes on SIMULINK model: (i) the block that determines if the puck is inside the wall is a user defined function block, (ii) use a fixed step solver (either `ode4` or `ode5`) with step size *Deltat*, (iii) use an initial condition zero for the integrator from acceleration to velocity, and z_0 for the integrator from velocity to position.

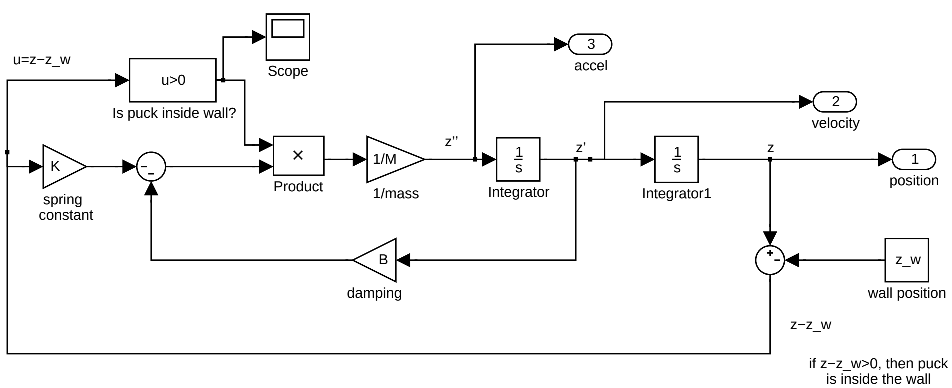

Figure 7: SIMULINK Model of Virtual Wall with Spring and Damping