Motor Control

- Suppose we wish to use a microprocessor to control a motor
  - (or to control the load attached to the motor!)

- Convert discrete signal to analog voltage
  - D/A converter
  - pulse width modulation (PWM)

- Amplify the analog signal
  - power supply
  - amplifier

- Types of power amplifiers
  - linear vs. PWM
  - voltage-voltage vs. transconductance (voltage-current)

- DC Motor
  - How does it work?

- What to control?
  - electrical signals: voltage, current
  - mechanical signals: torque, speed, position

- Sensors: Can we measure the signal we wish to control (feedback control)?
Outline

• Review of Motor Principles
  - torque vs. speed
  - voltage vs current control
  - with and without load

• D/A conversion vs. PWM generation
  - harmonics
  - advantages and disadvantages
  - creating PWM signals

• power amplifiers
  - linear vs PWM
  - voltage vs transconductance

• Control
  - choice of signal to control
  - open loop
  - feedback

• References are [5], [3], [1], [4], [8], [7], [6], [9]
Motor Review

- Recall circuit model of motor:

\[
V_B = K_V \Omega
\]

- Suppose motor is driven by a constant voltage source. Then steady state speed and torque satisfy

\[
\Omega = \frac{K_M V - R T_L}{K_M K_V + R B}
\]

\[
T_M = \frac{K_M (V B + K_V T_L)}{K_M K_V + R B}
\]

- Torque-speed curve

\[
T_M \quad \text{increasing } \Omega
\]

\[
\text{increasing } V
\]
Voltage Control

• Suppose we attempt to control speed by driving motor with a constant voltage.

• With no load and no friction \((T_L = 0, \ B = 0)\)

\[
\Omega = \frac{V}{K_V}
\]

\[
T_M = 0
\]

• Recall that torque is proportional to current: \(T_M = K_M I\). Hence, with no load and no friction, \(I = 0\), and motor draws no current in steady state.

• Current satisfies

\[
I = \frac{V - V_B}{R}
\]

• In steady state, back EMF balances applied voltage, and thus current and motor torque are zero.

• With a load or friction, \((T_L \neq 0 \text{ and/or } B \neq 0)\)

\[
\Omega < \frac{V}{K_V}
\]

\[
T_M > 0
\]

• Speed and torque depend on load and friction
  - friction always present (given in part by motor spec, but there will be additional unknown friction)
  - load torque may also be unknown, or imprecisely known
**Issue: Open Loop vs Feedback Control**

- Using constant voltage control we cannot specify desired torque or speed precisely due to friction and load - an *open loop* control strategy
- can be resolved by adding a sensor and applying *closed loop*, or *feedback* control

• add a tachometer for speed control

- add a current sensor for torque \( T_M = K_M I \) control

• Will study feedback control in Lecture 7.
Issue: Steady State vs. Transient Response

- Steady state response: the response of the motor to a constant voltage input eventually settles to a constant value
  - the torque-speed curves give steady-state information
- Transient response: the preliminary response before steady state is achieved.
- The transient response is important because
  - transient values of current, voltage, speed, . . . may become too large
  - transient response also important when studying response to nonconstant inputs (sine waves, PWM signals)
- The appropriate tool for studying transient response of the DC motor (or any system) is the transfer function of the system
System

- A system is any object that has one or more inputs and outputs

- Input: applied voltage, current, foot on gas pedal, . . .
- Output: other variable that responds to the input, e.g., voltage, current, speed, torque, . . .

- Examples:
  - RC circuit

  \[ R \] \[ C \] \[ V_i(t) \] \[ V_o(t) \]

  Input: applied voltage, Output: voltage across capacitor

  - DC motor

  \[ V \] \[ V_B = K_v \Omega \] \[ T_M, \Omega \] \[ T_L \] \[ J \]

  Input: applied voltage, Output: current, torque, speed
Stability

- We say that a system is *stable* if a bounded input yields a bounded output
- If not, the system is *unstable*
- Consider DC Motor with no retarding torque or friction
  - With constant voltage input, the steady state shaft speed $\Omega$ is constant $\Rightarrow$ the system from $V$ to $\Omega$ is stable
  - Suppose that we could hold current constant, so that the steady state torque is constant. Since
    \[ \frac{d\Omega}{dt} = \frac{T_M}{J}, \]
    the shaft velocity $\Omega \to \infty$ and velocity increases without bound $\Rightarrow$ the system from $I$ to $\Omega$ is unstable

- Tests for stability
  - mathematics beyond scope of class
  - we will point out in examples how stability depends on system parameters
Frequency Response

- A linear system has a *frequency response* function that governs its response to inputs:

\[ H(j\omega) \]

- If the system is *stable*, then the steady state response to a sinusoidal input, \( u(t) = \sin(\omega t) \), is given by \( H(j\omega) \):

\[ y(t) \rightarrow |H(j\omega)| \sin(\omega t + \angle H(j\omega)) \]

- We have seen this idea in Lecture 2 when we discussed anti-aliasing filters and RC circuits

- The response to a constant, or step, input, \( u(t) = u_0, t \geq 0 \), is given by the DC value of the frequency response:

\[ y(t) \rightarrow H(0)u_0 \]
Bode Plot Example

Lowpass filter\(^1\), \(H(j\omega) = 1/(j\omega + 1)\)

Steady state response to input \(\sin(10t)\) satisfies \(y_{ss}(t) = 0.1\sin(10t - 85^\circ)\).

\(^1\)MATLAB file `bode_plot.m`
**Frequency Response and the Transfer Function**

- To compute the frequency response of a system in MATLAB, we must use the *transfer function* of the system.

- (under appropriate conditions) a time signal $v(t)$ has a Laplace transform

  $$V(s) = \int_0^\infty v(t)e^{-st}dt$$

- Suppose we have a system with input $u(t)$ and output $y(t)$

  $$\begin{array}{c}
  u(t) \rightarrow H(s) \rightarrow y(t)
  \end{array}$$

- The transfer function relates the Laplace transform of the system output to that of its input:

  $$Y(s) = H(s)U(s)$$

- for simple systems $H(s)$ may be computed from the differential equation describing the system

- for more complicated systems, $H(s)$ may be computed from rules for combining transfer functions

- To find the frequency response of the system, set $s = j\omega$, and obtain $H(j\omega)$
Transfer Function of an RC Circuit

- **RC circuit**
  - Input: applied voltage, \( v_i(t) \).
  - Output: voltage across capacitor, \( v_o(t) \)

\[
\begin{array}{c}
\text{R} \\
\downarrow \\
v_i(t) \\
+ \\
\downarrow \\
C \\
\downarrow \\
v_o(t)
\end{array}
\]

- differential equation for circuit
  - Kirchhoff’s Laws: \( v_i(t) - I(t)R = v_o(t) \)
  - current/voltage relation for capacitor: \( I(t) = C \frac{dv_o(t)}{dt} \)
  - combining yields

\[
RC \frac{dv_o(t)}{dt} + v_o(t) = v_i(t)
\]

- To obtain transfer function, replace
  - each time signal by its Laplace transform: \( v(t) \rightarrow V(s) \)
  - each derivative by “\( s \)” times its transform: \( \frac{dv(t)}{dt} \rightarrow sV(s) \)
  - solve for \( V_o(s) \) in terms of \( V_i(s) \):

\[
V_o(s) = H(s)V_i(s), \quad H(s) = \frac{1}{RCs + 1}
\]

- To obtain frequency response, replace \( j\omega \rightarrow s \)

\[
H(j\omega) = \frac{1}{RCj\omega + 1}
\]
Transfer Functions and Differential Equations

- Suppose that the input and output of a system are related by a differential equation:

\[
\frac{d^ny}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + a_2\frac{d^{n-2}y}{dt^{n-2}} + \ldots + a_{n-1}\frac{dy}{dt} + a_ny = \\
b_1\frac{d^{n-1}u}{dt^{n-1}} + b_2\frac{d^{n-2}u}{dt^{n-2}} + \ldots + b_{n-1}\frac{du}{dt} + b_nu
\]

- Replace \( d^m y / dt^m \) with \( s^m Y(s) \):

\[
\left( s^n + a_1s^{n-1} + a_2s^{n-2} + \ldots + a_{n-1}s + a_n \right) Y(s) = \\
\left( b_1s^{n-1} + b_2s^{n-2} + \ldots + b_{n-1}s + b_n \right) U(s)
\]

- Solving for \( Y(s) \) in terms of \( U(s) \) yields the transfer function as a ratio of polynomials:

\[
Y(s) = H(s)U(s), \quad H(s) = \frac{N(s)}{D(s)}
\]

\[
N(s) = b_1s^{n-1} + b_2s^{n-2} + \ldots + b_{n-1}s + b_n
\]

\[
D(s) = s^n + a_1s^{n-1} + a_2s^{n-2} + \ldots + a_{n-1}s + a_n
\]

- The transfer function governs the response of the output to the input with all initial conditions set to zero.
Combining Transfer Functions

- There are (easily derivable) rules for combining transfer functions

  - Series: a series combination of transfer functions

    \[
    \begin{align*}
    \text{u(t)} & \xrightarrow{G(s)} \text{H(s)} & \text{y(t)} \\
    \end{align*}
    \]

    reduces to

    \[
    \begin{align*}
    \text{u(t)} & \xrightarrow{G(s)H(s)} \text{y(t)} \\
    \end{align*}
    \]

  - Parallel: a parallel combination of transfer functions

    \[
    \begin{align*}
    \text{H(s)} & \text{y(t)} \\
    \text{G(s)} & \text{y(t)} \\
    \Sigma & \\
    \end{align*}
    \]

    reduces to

    \[
    \begin{align*}
    \text{u(t)} & \xrightarrow{G(s)+H(s)} \text{y(t)} \\
    \end{align*}
    \]
Feedback Connection

• Consider the feedback system

Feedback equations: the output depends on the error, which in turn depends upon the output!
(a) \( y = Ge \)
(b) \( e = u \mp Hy \)

If we use “negative feedback”, and \( H = 1 \), then \( e = y - u \)
- the input signal \( u \) is a “command” to the output signal \( y 
- \( e \) is the error between the command and the output

• Substituting (b) into (a) and solving for \( y \) yields

• The error signal satisfies
Motor Transfer Function, I

- Four different equations that govern motor response, and their transfer functions

  - Current: Kirchoff’s Laws imply
    \[ L \frac{dI}{dt} + RI = V - V_B \]
    \[ I(s) = \left( \frac{1}{sL + R} \right) (V(s) - V_B(s)) \]  \hspace{1cm} (1)

  - Speed: Newton’s Laws imply
    \[ J \frac{d\Omega}{dt} = T_M - B\Omega - T_L \]
    \[ \Omega(s) = \left( \frac{1}{sJ + B} \right) (T_M(s) - T_L(s)) \]  \hspace{1cm} (2)

  - Torque:
    \[ T_M(s) = K_M I(s) \]  \hspace{1cm} (3)

  - Back EMF:
    \[ V_B(s) = K_V \Omega(s) \]  \hspace{1cm} (4)

⇒ We can solve for the outputs \( T_M(s) \) and \( \Omega(s) \) in terms of the inputs \( V(s) \) and \( T_L(s) \)
Motor Transfer Function, II

• Combine (1)-(4):

\[
\begin{align*}
\Omega(s) &= \frac{KM}{sJ + B} \left( \frac{1}{sL + R} \right) \left( V(s) - V_B(s) \right) \\
&= \left( \frac{KM}{(sL + R)(sJ + B) + KMKV} \right) V(s)
\end{align*}
\]

• Transfer function from Voltage to Speed (set \( T_L = 0 \)):
  - First combine (1)-(3)
  - Then substitute (4) and solve for \( \Omega(s) = H(s)V(s) \):

\[
\begin{align*}
\Omega(s) &= \frac{KM}{sJ + B} \left( \frac{1}{sL + R} \right) \left( V(s) - V_B(s) \right) \\
&= \left( \frac{KM}{(sL + R)(sJ + B) + KMKV} \right) V(s)
\end{align*}
\]

• Similarly, \( T_M(s) = \frac{KM(sJ + B)}{(sL + R)(sJ + B) + KMKV} V(s) \)

• The steady state response of speed and torque to a constant voltage input \( V \) is obtained by setting \( s = 0 \) (cf. Lecture 5):

\[
\begin{align*}
\Omega_{ss} &= \frac{KMV}{RB + KMKV}, \\
T_{Mss} &= \frac{KM BV}{RB + KMKV}
\end{align*}
\]
Motor Frequency Response

- DC Motor is a lowpass filter\(^2\)

\begin{figure}
\centering
\includegraphics[width=\textwidth]{dc_motor_freq_response.png}
\caption{DC motor frequency response}
\end{figure}

- Parameter Values
  - \( K_M = 1 \text{ N-m/A} \)
  - \( K_V = 1 \text{ V/(rad/sec)} \)
  - \( R = 10 \text{ ohm} \)
  - \( L = 0.01 \text{ H} \)
  - \( J = 0.1 \text{ N-m/(rad/sec)}^2 \)
  - \( B = 0.28 \text{ N-m/(rad/sec)} \)

- Why is frequency response important?
  - Linear vs. PWM amplifiers . . .

\(^2\)Matlab m-file DC\_motor\_freq\_response.m
Linear Power Amplifier

- Voltage amplifiers:

  ![Voltage Amplifier Diagram]

  - output voltage is a scaled version of the input voltage, gain measured in $V/V$.
  - Draws whatever current is necessary to maintain desired voltage
  - Motor speed will depend on load: $\Omega = \frac{K_M V - R T_L}{K_M K_V + R_B}$

- Current (transconductance) amplifiers:

  ![Current Amplifier Diagram]

  - output current is a scaled version of the input voltage, gain measured in $A/V$.
  - Will produce whatever output voltage is necessary to maintain desired current
  - Motor torque will not depend on load: $T_M = K_M I$

- Advantage of linearity: Ideally, the output signal is a constant gain times the input signal, with no distortion
  - In reality, bandwidth is limited
  - Voltage and/or current saturation

- Disadvantage:
  - inefficient unless operating “full on”, hence tend to consume power and generate heat.
Pulse Width Modulation

- Recall:
  - with no load, steady state motor speed is proportional to applied voltage
  - steady state motor torque is proportional to current (even with a load)
- With a D/A converter and linear amplifier, we regulate the level of applied voltage (or current) and thus regulate the speed (or torque) of the motor.
- PWM idea: Apply full scale voltage, but turn it on and off periodically
  - Speed (or torque) is (approximately) proportional to the average time that the voltage or current is on.
- PWM parameters:
  - switching period, seconds
  - switching frequency, Hz
  - duty cycle, %
- see the references plus the web page [2]
PWM Examples

- **40% duty cycle**: 

  ![PWM Signal 40% Duty Cycle](image)

- **10% duty cycle**: 

  ![PWM Signal 10% Duty Cycle](image)

- **90% duty cycle**: 

  ![PWM Signal 90% Duty Cycle](image)

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Matlab files PWM.plots.m and PWM.mdl

EECS461, Lecture 6, updated September 23, 2008
PWM Frequency Response, I

- Frequency spectrum of a PWM signal will contain components at frequencies \( k/T \) Hz, where \( T \) is the switching period.
- PWM input: switching frequency 10 Hz, duty cycle 40\%:\n
![PWM signal graph]

- Frequency spectrum will contain
  - a nonzero DC component (because the average is nonzero)
  - components at multiples of 10 Hz

![Frequency spectrum graph]

\(^4\) Matlab files PWM_spectrum.m and PWM.mdl
PWM Frequency Response, II

- PWM signal with switching frequency 10 Hz, and duty cycle for the $k$'th period equal to $0.5(1 + \cos(0.2\pi kT))$ (a 0.1 Hz cosine shifted to lie between 0 and 1, and evaluated at the switching times $T = 0.1$ sec)$^5$

- Remove the DC term by subtracting 0.5 from the PWM signal

---

$^5$Matlab files PWM_sinusoid.m and PWM.mdl
PWM Frequency Response, III

- Frequency spectrum of PWM signal has
  - zero DC component
  - components at $\pm 0.1$ Hz
  - components at multiples of the switching frequency, 10 Hz

![Frequency response of PWM signal](image)

- Potential problem with PWM control:
  - High frequencies in PWM signal may produce undesirable oscillations in the motor (or whatever device is driven by the amplified PWM signal)
  - switching frequency usually set $\approx 25$ kHz so that switching is not audible
PWM Frequency Response, IV

- Suppose we apply the PWM output to a lowpass filter that has unity gain at 0.1 Hz, and small gain at 10 Hz

Then, after an initial transient, the filter output has a 0.1 Hz oscillation.
PWM Generation

- Generate PWM using D/A and pass it through a PWM amplifier

![Diagram of PWM generation](image)

- techniques for generating analog PWM output ([6]):
  - software
  - timers
  - special modules
- Feed the digital information directly to PWM amplifier, and thus bypass the D/A stage

![Diagram of direct PWM](image)

- PWM voltage or current amplifiers
- must determine direction
  - normalize so that
    * 50% duty cycle represents 0
    * 100% duty cycle represents full scale
    * 0% duty cycle represents negative full scale
    * what we do in lab, plus we limit duty cycle to 35% − 65%
  - use full scale, but keep track of sign separately
References


