EECS 477: Introduction to algorithms.
Lecture 2
TTh 8:30-10am

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Lecture outline

• Mathematical induction
  – Simple case
  – Generalized induction
  – Common mistakes

• Limits, De l’Hôpital rule
Mathematical induction

- Mathematical induction: a rigorous proof procedure

- Need to prove $P(n), \forall n \in \mathbb{N}, n \geq a$

- Two stages
  1. Basis: $P(a)$
  2. Induction step: $(\forall n > a)[P(n-1) \Rightarrow P(n)]$

- Example: $\sum_{i=1}^{n}(2i - 1) = n^2$
Generalized mathematical induction

• Need to prove $P(n), \forall n \in \mathbb{N}, n \geq a$

• Two stages

  1. Basis: $P(n_0)$ holds for all $n_0 \in [a..b - 1]$.

  2. Induction step: for any $n \geq b$ we have

     $$(\forall k \in [a..n - 1])[P(k)] \Rightarrow P(n)$$

• Example: every positive integer can be expressed as a product of prime numbers.
Mathematical induction: a la russe I

```c
int russe(int m, int n) {
    if(m==1)
        return n;
    else {
        if(m%2==0)
            return russe(m/2,n*2);
        else
            return n + russe(m/2, n*2);
    }
}
```

We can prove the correctness of the algorithm using induction on \( m \).
Mathematical induction: a la russe II

- Basis: \( m = 1 \)

- Induction: suppose that \( \text{russe}(s, t) \) returns \( s \times t \) for any \( s < m \). Need to prove that \( \text{russe}(m, n) \) returns \( m \times n \). Two cases to consider:

  even: \( m = 2 \times t \)

  odd: \( m = 2 \times t + 1 \)
Mathematical induction: miscellaneous

- It is sometimes easier to prove a stronger statement
  - Prove that $\sum_{i=1}^{n}(2i - 1)$ is a square.

- Common pitfalls
  - No basis
  - Wrong induction step: horse of a different color (1.6.2)
  - $2^n > n^2$
Limits I

• $f(n)$ tends to the limit $a$ as $n$ tends to infinity if for any positive real number $\epsilon > 0$ there is $N(\epsilon)$ such that

$$|f(n) - a| < \epsilon,$$

for all $n > N(\epsilon)$.

we then say $\lim_{n \to \infty} f(n) = a$ or $f(n) \to a$ as $n \to \infty$.

• Example: $\lim_{n \to \infty} r^n$ for $|r| < 1$.

• Indeed $N(\epsilon) = \lceil \log_{|r|} \epsilon \rceil$ and monotonicity.
Limits II

• \( f(n) \) tends to \(+\infty\) as \( n \) tends to infinity if for any positive real number \( A > 0 \) there is \( N(A) \) such that

\[
f(n) > A, \text{ for all } n > N(A)
\]

we then say \( \lim_{n \to \infty} f(n) = +\infty \) or \( f(n) \to +\infty \) as \( n \to \infty \).

• Example: \( \lim_{n \to \infty} R^n \) for \( R > 1 \).

• Indeed \( N(A) = \lceil \log_R A \rceil \) and monotonicity.
Limits: simple properties

• \( \lim_{n \to \infty} (f(n) + g(n)) = \lim_{n \to \infty} f(n) + \lim_{n \to \infty} g(n) \) if both RHS limits exist;

• \( \lim_{n \to \infty} (f(n)g(n)) = \lim_{n \to \infty} f(n) \lim_{n \to \infty} g(n) \) if both RHS limits exist;

• \( \lim_{n \to \infty} (f(n)/g(n)) = \lim_{n \to \infty} f(n)/\lim_{n \to \infty} g(n) \) if both RHS limits exist and \( \lim_{n \to \infty} g(n) \neq 0 \);
Limits example

• Solve applying facts from the previous slide:

$$\lim_{n \to \infty} \frac{1 + 2^n}{2^n} = ?$$

• What about this one:

$$\lim_{n \to \infty} \frac{\log n}{n} = ?$$

Here we shall use $\log x := \log_2 x = \log_2 e \ln x = \ln x / \ln 2$. 
De l’Hôpital rule

• Suppose that \( \lim_{n \to \infty} f(n) = \lim_{n \to \infty} g(n) = 0 \) (or \( +\infty \)). Suppose we can construct extensions of \( f \) and \( g \) to the real line. Then

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = \frac{\lim_{x \to \infty} f'(x)}{\lim_{x \to \infty} g'(x)}
\]

if both of the RHS limits exist and the RHS denominator is not zero.

• What about this one:

\[
\lim_{n \to \infty} \frac{\log n}{n} = \frac{\lim_{x \to \infty} [1/(x \ln 2)]}{\lim_{x \to \infty} 1} = 0/1 = 0.
\]
More things to know

- Geometric series

- Combinatorics, permutations, factorial, \((1 + x)^n\)