EECS 477: Introduction to algorithms.

Lecture 2

TTh 8:30-10am

Prof. Igor Guskov
guskov@eecs.umich.edu
office: 126 ATL building (AI lab)

September 5, 2002

Lecture outline

- Mathematical induction
  - Simple case
  - Generalized induction
  - Common mistakes
- Limits, De l'Hôpital rule

Mathematical induction

- Mathematical induction: a rigorous proof procedure

- Need to prove $P(n), \forall n \in \mathbf{N}, n \geq a$

- Two stages

  1. Basis: $P(a)$

  2. Induction step: $(\forall n > a)[P(n-1) \Rightarrow P(n)]$

- Example: $\sum_{i=1}^{n}(2i-1) = n^2$

Generalized mathematical induction

- Need to prove $P(n), \forall n \in \mathbf{N}, n \geq a$

- Two stages

  1. Basis: $P(n_0)$ holds for all $n_0 \in [a..b-1]$.

  2. Induction step: for any $n \geq b$ we have

$$(\forall k \in [a..n-1])[P(k)] \Rightarrow P(n)$$

- Example: every positive integer can be expressed as a product of prime numbers.

Mathematical induction: a la russe I

```
int russe(int m, int n) {
  if(m==1)
    return n;
  else {
    if(m%2==0)
      return russe(m/2,n*2);
    else
      return n + russe(m/2, n*2);
  }
}
```

We can prove the correctness of the algorithm using induction on m.

Mathematical induction: a la russe II

- Basis: $m = 1$

- Induction: suppose that $russe(s, t)$ returns $s * t$ for any $s < m$. Need to prove that $russe(m, n)$ returns $m * n$. Two cases to consider:

even: $m = 2 * t$

odd: $m = 2 * t + 1$:

Mathematical induction: miscellaneous

- It is sometimes easier to prove a stronger statement
  - Prove that $\sum_{i=1}^{n}(2i-1)$ is a square.

- Common pitfalls
  - No basis
  - Wrong induction step: horse of a different color (1.6.2)
  - $2^n > n^2$

ϵ

Limits I

- $f(n)$ tends to the limit a as n tends to infinity if for any positive real number $\epsilon > 0$ there is $N(\epsilon)$ such that

$$|f(n) - a| < \epsilon, \text{for all } n > N(\epsilon)$$

  we then say $\lim_{n\to\infty} f(n) = a$ or $f(n) \to a$ as $n \to \infty$.

- Example: $\lim_{n\to\infty} r^n$ for $|r| < 1$.

- Indeed $N(\epsilon) = \left\lceil \log_{|r|} \epsilon \right\rceil$ and monotonicity.

Limits II

- $f(n)$ tends to $+\infty$ as n tends to infinity if for any positive real number $A > 0$ there is $N(A)$ such that

$$f(n) > A, \text{for all } n > N(A)$$

  we then say $\lim_{n\to\infty} f(n) = +\infty$ or $f(n) \to +\infty$ as $n \to \infty$.

- Example: $\lim_{n\to\infty} R^n$ for $R > 1$.

- Indeed $N(A) = \lceil \log_R A \rceil$ and monotonicity.

Limits: simple properties

- $\lim_{n\to\infty}(f(n) + g(n)) = \lim_{n\to\infty} f(n) + \lim_{n\to\infty} g(n)$ if both RHS limits exist;

- $\lim_{n\to\infty}(f(n)g(n)) = \lim_{n\to\infty} f(n) \lim_{n\to\infty} g(n)$ if both RHS limits exist;

- $\lim_{n\to\infty}(f(n)/g(n)) = \lim_{n\to\infty} f(n)/ \lim_{n\to\infty} g(n)$ if both RHS limits exist and $\lim_{n\to\infty} g(n) \neq 0$;

Limits example

- Solve applying facts from the previous slide:

$$\lim_{n\to\infty} \frac{1+2^n}{2^n} = ?$$

- What about this one:

$$\lim_{n\to\infty} \frac{\log n}{n} = ?$$

  Here we shall use $\log x := \log_2 x = \log_2 e \ln x = \ln x / \ln 2$.

De l'Hôpital rule

- Suppose that $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = 0$ (or $+\infty$). Suppose we can construct extensions of f and g to the real line. Then
$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{\lim_{x\to\infty} f'(x)}{\lim_{x\to\infty} g'(x)}$$
if both of the RHS limits exist and the RHS denominator is not zero.

- What about this one:
$$\lim_{n\to\infty} \frac{\log n}{n} = \frac{\lim_{x\to\infty}[1/(x \ln 2)]}{\lim_{x\to\infty} 1} = 0/1 = 0.$$

More things to know

- Geometric series
- Combinatorics, permutations, factorial, $(1 + x)^n$