EECS 477: Introduction to algorithms.

Lecture 12

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Lecture outline: greedy algorithms II

- Knapsack
- Scheduling
  - minimizing time
  - with deadlines

Greedy algorithms

- Problem:
  - Optimize cost (w.r.t. objective function)
  - Solutions are composed from components (candidates)
  - A *validity check* function
  - A *feasibility check* function (checks whether the partial solution can be completed to a valid solution)
- Greedy algorithm
  - goes step by step
  - maintains partial solution and possible extensions
  - *selects* the best possible extensions (how?)

Greedy verbatim

```
Greedy() {
    initialize_partial_solution();
    while(there are extensions) {
        do {
            choose the best extension;
            check its validity;
        while(it's not valid);
        add that extension to get a new partial solution
        see if the problem is solved, if so return;
    }
}
```

Knapsack

- n objects, weights w_i, values v_i
- a knapsack can hold maximal weight of W
- maximize the $ value of what we fit

$\boxed{\text{(2lb, \$5)}}$ $\boxed{\text{(3lb, \$2)}}$ $\boxed{\text{(5lb, \$100)}}$ $\boxed{\text{(12lb, \$6)}}$

$\boxed{\text{max of 9lb}}$

Knapsack: much simpler case?

- or is it?
- n objects, weights w_i, values v_i, $w_i = v_i$
- a knapsack can hold maximal weight of $W = \frac{1}{2}\sum_{i=1}^{n} w_i$
- can we fill it up full?
- No polynomial time algorithm is known!

(2lb) (3lb) (5lb) (12lb)

max of 11lb

Knapsack: breakable objects

- n objects, weights w_i, values v_i
- a knapsack can hold maximal weight of W
- we can break objects, fractions $x_i \in (0,1)$
- maximize $\sum_{i=1}^{n} x_i v_i$ subject to $\sum_{i=1}^{n} x_i w_i \leq W$

$x_1 = 1$	$x_2 = 2/3$	$x_3 = 1$	$x_4 = 0$
(2lb, \$5)	(3lb, \$2)	(5lb, \$100)	(12lb, \$6)

max of 9lb

Knapsack verbatim: greedy algorithm

```
float knapsack(const vector<float>& w, const vector<float>& v,
               vector<float>& x, float wmax) {
    for(int i=0; i<x.size(); ++i) x[i] = 0;
    float weight = 0, value = 0;
    while(weight<wmax) {
        i = best_remaining_object();
        if(weight+w[i]<=wmax) {
            x[i] = 1; weight+=w[i]; value+=v[i];
        } else {
            x[i] = (wmax-weight)/w[i]; weight+=w[i]; value+=v[i];
            return value;
        }
    }
    return value;
}
```

Knapsack: strategies

- Choose the most valuable
- Sequence: 3, 4.

$x_1 = 0$	$x_2 = 0$	$x_3 = 1$	$x_4 = 1/3$
(2lb, $5)	(3lb, $2)	(5lb, $100)	(12lb, $6)

max of 9lb, value $102

Knapsack: strategies

- Choose the lightest
- Sequence: 1, 2, 3.

$x_1 = 1$	$x_2 = 1$	$x_3 = 4/5$	$x_4 = 0$
(2lb, \$5)	(3lb, \$2)	(5lb, \$100)	(12lb, \$6)

max of 9lb, value \$87

Knapsack: strategies

- Choose the best value per unit weight (that is v_i/w_i)
- Sequence: 3, 1, 2.
- Provably the best strategy

$x_1 = 1$	$x_2 = 2/3$	$x_3 = 1$	$x_4 = 0$
(2lb, \$5)	(3lb, \$2)	(5lb, \$100)	(12lb, \$6)

max of 9lb, value \$106.33

Knapsack: greedy best theorem

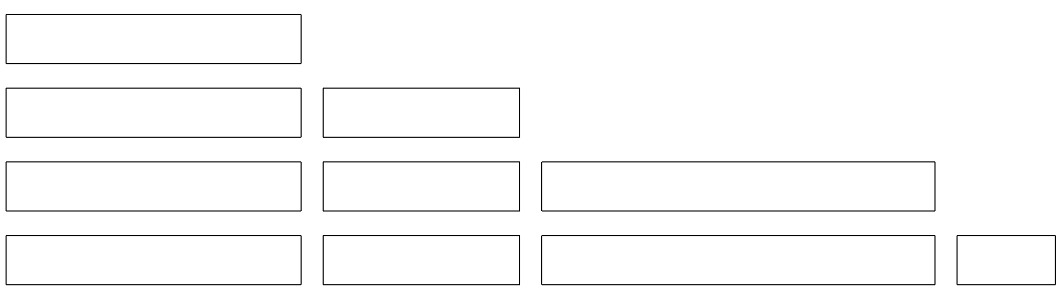
- Choose the best value per unit weight (that is v_i/w_i)
- **Theorem:** If objects are selected in order of decreasing v_i/w_i, then the algorithm knapsack finds an optimal solution.
- **Proof outline:** if all the weights are 1 then it is optimal. Otherwise, order them so that v_i/w_i decrease, and consider $V(x) = \sum_{i=1}^{n} x_i v_i$.
- For some other choice y prove that (choose special j)

$$V(x) - V(y) = \sum_{i=1}^{n} (x_i - y_i) v_i = \sum_{i=1}^{n} (x_i - y_i) w_i \frac{v_i}{w_i} \geq 0.$$

$$V(x) - V(y) \geq \sum_{i=1}^{n} (x_i - y_i) w_i \frac{v_j}{w_j} = \frac{v_j}{w_j} \sum_{i=1}^{n} (x_i - y_i) w_i \geq 0.$$

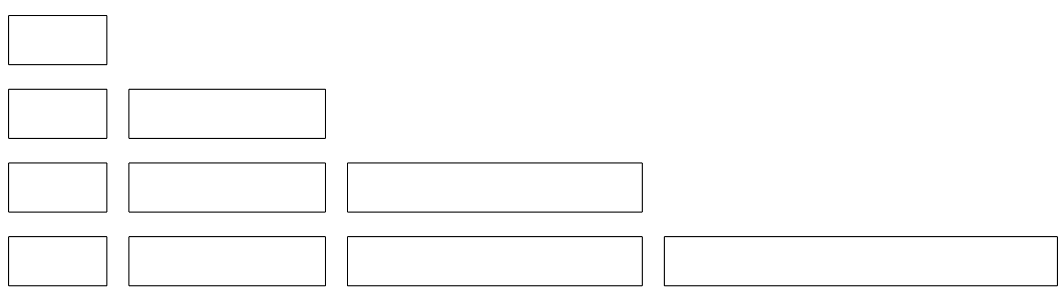
Scheduling: min time in system

- A single server
- n customers
- service times t_i
- would like to minimize average time that a customer spends in the system
- since n is fixed we can just minimize $T(p) = \sum_{i=1}^{n} \sum_{k=1}^{i} t_{p_i}$.



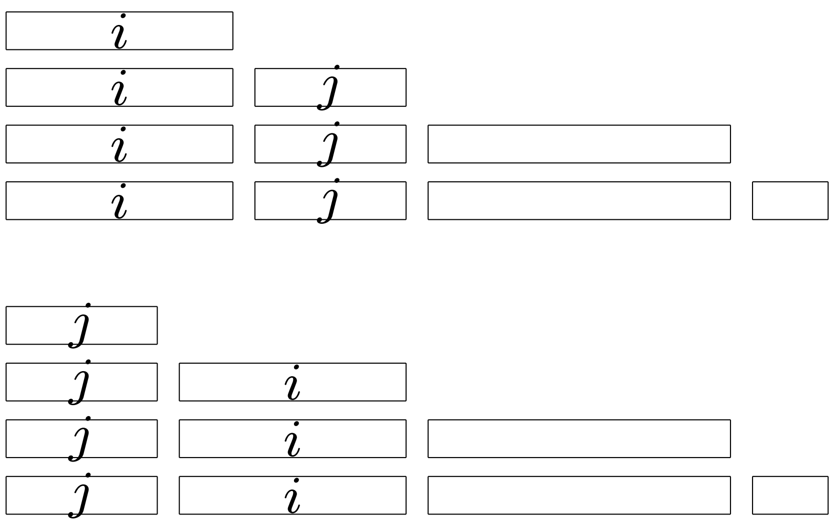
Scheduling: min time in system

- Claim: serving in order of increasing service time is optimal
- First of all, $T(p) = \sum_{i=1}^{n} \sum_{k=1}^{i} t_{p_k} = \sum_{k=1}^{n} (n-k+1) t_{p_k}$.
- p is a permutation



Scheduling: min time in system

- If not in order, then we find $i < j$ such that $t_{p_i} > t_{p_j}$
- Exchange them!
- Total cost will be better
- Implementation trivial: running time $O(n \log n)$



Scheduling with deadlines

- A single server
- n jobs, service time is the same (unit).
- At any time just one job
- Deadlines d_i: profit g_i is earned only if $t_i \leq d_i$.
- $t_i = \infty$ if it's not executed
- Example: $n = 5$

i	1	2	3	4	5
g_i	40	15	20	60	30
d_i	2	3	2	3	1

Scheduling with deadlines

- Greedy strategy: take the fattest job still available as long as the set stays feasible
- Feasible set allows at least one feasible sequence of execution
- **Lemma:** set of k jobs if feasible if and only if indexed in order of non-decreasing deadlines the sequence $1, \ldots, k$ is feasible.
- **Only If Proof:** if sequence is not feasible then $r > d_r$ for some r, but then there are at least r jobs whose deadlines are before or on $t = r - 1$, so that the set is not feasible.
- **Theorem:** greedy algorithm is optimal. Proof in the book (p.208)

Scheduling with deadlines: slow algorithm

```
void sequence(const vector<int>& d, vector<int>& j) {
    assert(j.empty());
    j.push_back(0);
    for(i=1; i<d.size(); ++i) {
        r = j.size()-1;
        while( r>=0 && d[j[r]]>max(d[i], r) )
            --r; // while usefully shiftable
        if(d[i]>r) { // r contains first non-shiftable
            j.push_back(-1);
            for(int m=j.size()-1; m>r; --m)
                j[m+1] = j[m];
            j[r+1] = i;
        }
    }
} // so, how slow is this?
```

Scheduling with deadlines faster

- The same algorithm
- Different feasibility check:
- Start with an empty schedule of length n
- Schedule a job i at time

$$t_i = \max\{k : k \le \min(d_i, n-1) \text{ and } k \text{ is free}\}$$

- To implement define $n_t = \max\{k : k \le t \text{ and } k \text{ is free}\}$
- Two *slots* in the same set if their n_t are the same
- Use disjoint sets to maintain "the first available slot before or on t"

Scheduling with deadlines: faster algorithm

- Algorithm is in the book p. 214
- Need to maintain array of earliest available times since labels of the disjoint set do not guarantee the minimality when merging them
  - find the desired time (deadline or n)
  - get its label l and the earliest available
  - if there is a slot insert yourself in there and merge set with label l and the one immediately to the left
  - finally, compress the solution
- at most $2n$ finds and n merges, so that without sorting we have $O(n\alpha(2n, n))$ where α is that slow growing function that is < 4.