P and NP, approximation

EECS 477

Lecture 22, 12/03/2002

Complexity classes

- P – class of decision problems that can be solved by a polynomial-time algorithm
- NP – non-deterministic polynomial time
  - Given a solution, it can be checked in polynomial time
    - given a cycle/tour – check?
    - Composite number: given a factor easy to check (but to find one?)

Theorems, conjectures

- Theorem: P $\subseteq$ NP
  - If we can solve a problem then we can surely check it
- The central open question:

  Is P=NP or not?
- Conjecture: P$\neq$NP
  - Look at the hardest problems in NP
    - Harder than any other problem in NP

Polynomial reduction

Polynomial reductions

- Two problems A and B
- A $\leq^p$ B :
  - A is polynomially reducible to B
  - There is an algorithm for solving A in time that would be polynomial if we could solve arbitrary instances of B in unit time
  - If both ways then they are polynomially equivalent: A $\equiv^p$ B
  - Transitive: if A $\leq^p$ B and B $\leq^p$ C then A $\leq^p$ C

Example

- HAM $\equiv^p$ HAMD
  - HAMD $\leq^p$ HAM
    - Trivial: if HAMalgo finds a cycle then yes
  - HAM $\leq^p$ HAMD
    - First check if HAMDalgo gives yes for the original graph
    - Start considering edges for removal one by one
      - Apply HAMDalgo to the remaining
      - If still Hamiltonian without an edge then remove it
      - Otherwise remove the edge and keep going
      - Stop when a cycle is left, return it

Reduction function

- Two decision problems $X \subseteq I$ and $Y \subseteq J$
- F: map $I \rightarrow J$

  such that $F(x) \in Y$ if and only if $x \in X$

  Theorem: If F is computable in polynomial time then $X \leq^p Y$

```
bool DecideX(x) {
    y = F(x);
    if(DecideY(y)) return true;
    else return false;
}
```

Example

- HAMD $\leq^p$ TSPD
  - Given a graph G=(N,A), need to see if it is Hamiltonian
  - Define F(G) be the TSPD instance with a complete graph (N,N$\times$N)
    - Cost = 1 if the edge in A and 2 otherwise
    - TSPD bound being N
    - If TSPD yes then that is a Hamiltonian cycle
    - If TSPD no then no Hamiltonian cycle

NP-completeness

- Decision problem X is NP-complete
  1. X is in NP
  2. Y $\leq^p$ X for every problem Y in NP
- X is polynomially harder than any other NP problem
- If we know that X is NP-complete and X $\leq^p$ Z then Z is NP-complete
- If we could only find one such X

SAT: satisfiability

- Given a boolean formula
  - Is it satisfiable? is there an assignment of values to variables that will make it true?
    - e.g. $(p \wedge q) \Rightarrow (p \vee q)$ is satisfiable via (p=q=true)
    - CNF: conjunctive normal form
  - SAT-CNF satisfiability for boolean expressions in CNF form
  - Cook's Theorem
    - For any NP problem Y, Y $\leq^p$ SAT-CNF

NP-completeness

- Now that we know one of NP-complete problems
  - We find a bunch of other ones
  - All we need to show is
    - that a problem X is in NP
    - and that SAT-CNF $\leq^p$ X
      - i.e. knowing that a polynomial algorithm for X would also mean that a polynomial algorithm exists for SAT-CNF
    - Then X is NP-complete

Some NP-complete problems

- SAT
- 3SAT: clauses have three variables
- 3DM: 3D matching
- HAMD: hamiltonian circuit
- PARTITION: set A and $s: A \rightarrow Z^+$
  - Partition A into two equally sized parts
- CLIQUE: clique of size J or more
- VERTEX COVER: of size K or less
- K-COL: graph colorability with K colors or less

SAT-3-CNF is NP-complete

- It is in NP
- Will prove SAT-CNF $\leq^p$ SAT-3-CNF
  - Given an instance of SAT-CNF construct an instance of SAT-3-CNF efficiently

  Consider one clause: disjunction of K literals

  K=3: done

  K=4: C=(L1+L2+L3+L4) is satisfiable if and only if

  C'=(L1+L2+U)($\neg$U+L3+L4) is satisfiable

  if C is true then C' is satisfiable,

  if C is false then no choice of U will make C' true.

SAT-3-CNF

- When K>4: C=(L[1]+…+L[K])

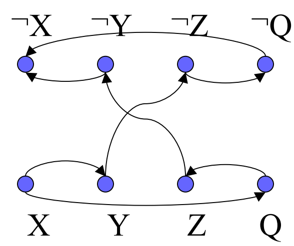
  C' =

  (L[1]+L[2]+U[1])(¬U[1]+L[3]+U[2])…(U[K-3]+L[K-1]+L[K])

  Again C is true iff C' is satisfiable
- Many clauses
  - Each with its own U's
    - For an instance of SAT-CNF built an instance of SAT-3-CNF

SAT-2-CNF

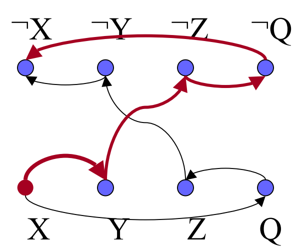
- Is in P
  - Formula (¬X+Y)(¬Z+¬Y)(¬Q+Z)(Q+¬X)
    - Construct directed graph: two verts for each variable, edge from X to Y if there is a clause equiv to (¬X+Y) = ¬(¬Y) + (¬X)



SAT-2-CNF

P

- Claim
  - If there are paths from some N to ¬N and from ¬N to N then the formula cannot be satisfied
  - Otherwise it is satisfiable (example)



X ⇒Y
Y ⇒ ¬Z
¬Z ⇒ ¬Q
¬Q ⇒ ¬X
Hence
X ⇒ ¬X
Hmm…

HAM vs Eulerian

- Hamiltonian path
  – directed graph G=(V,E) and two vertices s,t∈V
  – decide if there exists a path from s to t, which goes through each node once.
    - NP-complete (can construct graph for a SAT-3-CNF instance)
- Eulerian path
  – undirected graph G=(V,E) and two vertices s≠t∈V
  – decide if there exists a path from s to t, which goes through each edge exactly once.

A theorem

Theorem: A connected graph has an Eulerian path from s to t iff

1. s and t's degrees are odd.
2. the degrees of the other vertices are even

So EULER is in P.

Can construct path by a polynomial algorithm?

NP-hard

- X is NP-hard
  - if there is an NP-complete problem Y that can be polynomially reduced to X
    - $Y \leq^p X$
  - Does not have to be a decision problem
  - Decision problem can be NP-hard but not in NP, for instance exact K-colorability
    - Any K-coloring is a certificate for K-COL but not for K-COLE(exact: can color with K but no less)

What about two colors?

- Determine whether a graph is 2-vertex colorable
  - A polynomial algorithm?
    - DFS/BFS?

Metric TSP

- Undirected graph (complete)
- Distance matrix satisfies
  - Triangle inequality

    d(x,z) <= d(x,y) + d(y,z)
    - length(Hamiltonian cycle) >= length(Hamiltonian path) >= length(MST)
  - Construct an MST, tour around it will cost no more than 2*length(MST)
  - tour with shortcuts <= 2 length(MST)