



Transforms II

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Overview

- Homogeneous Coordinates
- 3-D Transforms
- Viewing Projections

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Homogeneous Coordinates

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$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \frac{x}{w} \\ \frac{y}{w} \\ \frac{z}{w} \end{bmatrix} \quad \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad \begin{bmatrix} \frac{x}{w} \\ \frac{y}{w} \\ \frac{z}{w} \\ 1 \end{bmatrix} \quad \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Allows translations to be included into matrix transform.

Allows us to distinguish between a vector and a point.

In perspective transformations the extra coordinate can be thought to contain the perspective information or scaling.

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Homogeneous Coordinates

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$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ w \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- As w gets smaller Real space X gets Larger
- When w reaches 0 X is now at infinity
- Homogeneous coordinates allows us to deal mathematically with infinity

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$$\begin{bmatrix} \frac{x}{w} \\ \frac{y}{w} \\ \frac{z}{w} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Point

$$\begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$$

Vector

Many different homogeneous space vectors for the same real space point

Vectors can not be translated but can be scaled and rotated

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Homogeneous Coordinates

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Attribute	Vector	Point
Represents	Magnitude and Direction	Location
Origin	Unique	Arbitrary
Transformation	Liner Scale and Rotate	Affine Move

$$\begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$$

Vector

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Point

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3-D Transforms

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Rotation

$$\begin{bmatrix} x''' \\ y''' \\ z''' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos q & -\sin q & 0 \\ 0 & \sin q & \cos q & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ z'' \\ 1 \end{bmatrix} \quad \begin{bmatrix} x'' \\ y'' \\ z'' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos f & 0 & \sin f & 0 \\ 0 & 1 & 0 & 0 \\ -\sin f & 0 & \cos f & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

Rotation about X

Rotation about Y

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos l & -\sin l & 0 & 0 \\ \sin l & \cos l & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotation about Z

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3-D Transforms

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Rotation

$$\begin{bmatrix} x''' \\ y''' \\ z''' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos q & -\sin q & 0 \\ 0 & \sin q & \cos q & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos f & 0 & \sin f & 0 \\ 0 & 1 & 0 & 0 \\ -\sin f & 0 & \cos f & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos l & -\sin l & 0 & 0 \\ \sin l & \cos l & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

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3-D Transforms

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Scale

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} scale_x & 0 & 0 & 0 \\ 0 & scale_y & 0 & 0 \\ 0 & 0 & scale_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

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3-D Transforms

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Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \Delta x \\ 0 & 1 & 0 & \Delta y \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

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3-D Transforms

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Composite Transform Matrix

$$[R] = [Roll][Pitch][Yaw]$$

$$[M] = [R][S][T]$$

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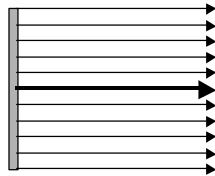


Viewing Projections

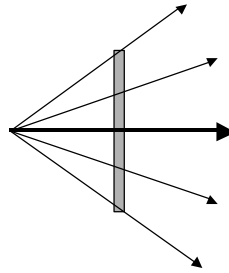
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How do we see in 3-D

Parallel Projections



Perspective Projections



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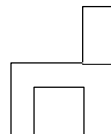
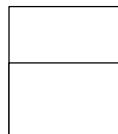
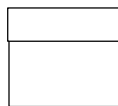
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Parallel Projections

Elevations:

Projection plane is perpendicular to a principle axis. Front, Top (Plan), Side.



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Viewing Projections

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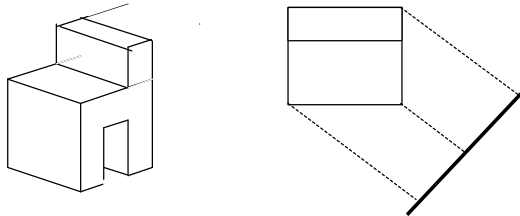
Parallel Projections

Axonometric:

Projection plane is not orthogonal to a principle axis.

Isometric:

Direction of projection makes equal angles with each principal axis.



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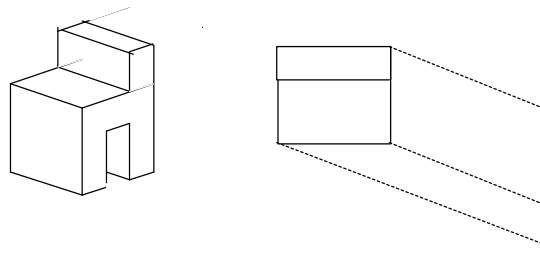
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Parallel Projections

Oblique:

Direction of projection is not orthogonal to the projection plane.



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Viewing Projections

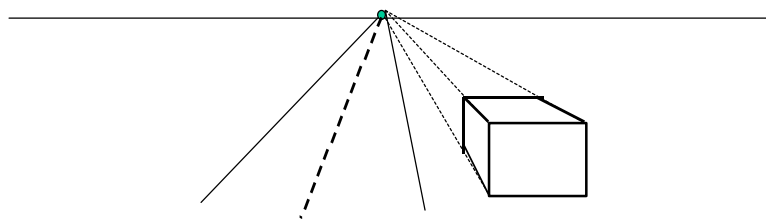
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Perspective Projection

One-point:

One principle axis cut by projection plane.

One axis vanishing point.



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Viewing Projections

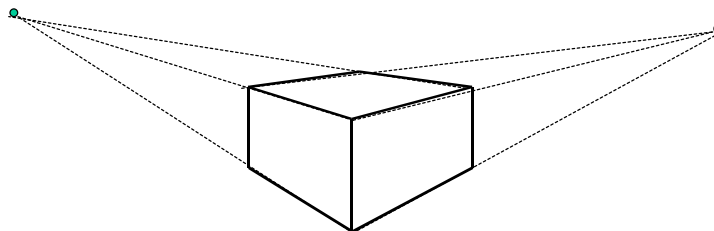
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Perspective Projection

Two-point:

Two principle axes cut by projection plane.

Two vanishing points.



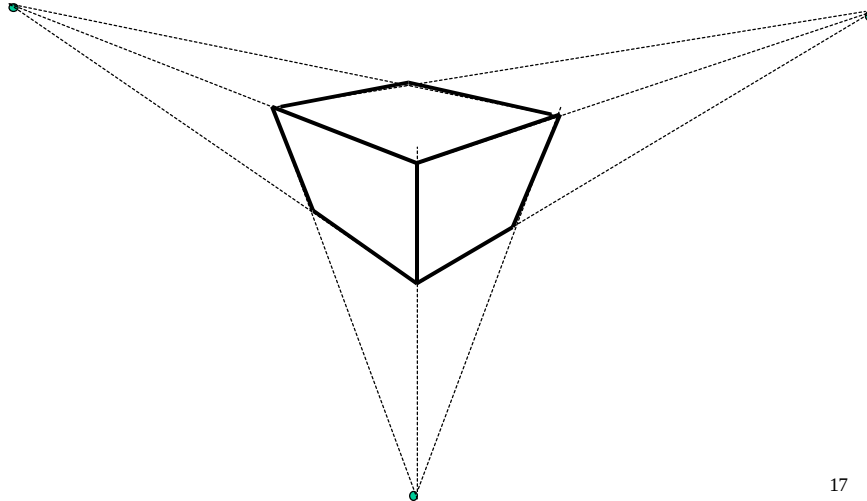
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Viewing Projections

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Perspective Projection Three-point



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Definitions

VRC	-	Viewing Reference Coordinate
VRP	-	View Reference Point
VPN	-	View Plane Normal
VUP	-	View Up Direction
DOP	-	Direction of Projection
PRP	-	Projection Reference Point
		Center of Projection
VP	-	Viewing Plane
BCP	-	Back Clipping Plane
FCP	-	Front Clipping Plane

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Parallel Projections

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1. Translate VRP to the origin
2. Rotate VRC such that n axis (VPN) becomes z, u axis becomes x, and v axis becomes y.
3. Shear such that the direction of projection becomes parallel to the z axis.
4. Translate and scale into the parallel-projection canonical view volume.

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Parallel Projections

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Step1:

Simply the negative of the VRP vector

$$\begin{bmatrix} 1 & 0 & 0 & -vrp_x \\ 0 & 1 & 0 & -vrp_y \\ 0 & 0 & 1 & -vrp_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Parallel Projections

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Step2:

$$\vec{R}_z = \frac{\vec{VPN}}{|\vec{VPN}|} \quad \vec{R}_x = \frac{\vec{VUP} \times \vec{R}_z}{|\vec{VUP} \times \vec{R}_z|} \quad \vec{R}_y = \vec{R}_z \times \vec{R}_x$$

$$R = \begin{bmatrix} R_{1x} & R_{2x} & R_{3x} & 0 \\ R_{1y} & R_{2y} & R_{3y} & 0 \\ R_{1z} & R_{2z} & R_{3z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Parallel Projections

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Step3:

$$CW = \begin{bmatrix} \frac{u_{\max} + u_{\min}}{2} \\ \frac{v_{\max} + v_{\min}}{2} \\ 0 \\ 1 \end{bmatrix} \quad PRP = \begin{bmatrix} prp_u \\ prp_v \\ prp_n \\ 1 \end{bmatrix} \quad DOP = \begin{bmatrix} dop_x \\ dop_y \\ dop_z \\ 0 \end{bmatrix} = CW - PRP$$

$$\begin{bmatrix} 1 & 0 & shear_x & 0 \\ 0 & 1 & shear_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$shear_x = -\frac{dop_x}{dop_z}$$

$$shear_y = -\frac{dop_y}{dop_z}$$

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Parallel Projections

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Step4:

Translation

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{u_{\max} + u_{\min}}{2} \\ 0 & 1 & 0 & -\frac{v_{\max} + v_{\min}}{2} \\ 0 & 0 & 1 & -F \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} \frac{2}{u_{\max} - u_{\min}} & 0 & 0 & 0 \\ 0 & \frac{2}{v_{\max} - v_{\min}} & 0 & 0 \\ 0 & 0 & \frac{1}{F - B} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Perspective Projections

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1. Translate VRP to the origin
2. Rotate VRC such that n axis (VPN) becomes z, u axis becomes x, and v axis becomes y.
3. Translate such that the center of Projection (COP), given by the PRP, is at the origin.
4. Shear such that the center line fo the view volume becomes the z axis.
5. Scale such that the view volume becomes the canonical perspective view volume, the truncated right pyramid defined by the six planes

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Perspective Projections

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Step1:

Simply the negative of the VRP vector

$$\begin{bmatrix} 1 & 0 & 0 & -vrp_x \\ 0 & 1 & 0 & -vrp_y \\ 0 & 0 & 1 & -vrp_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step2:

$$R_z = \frac{VPN}{|VPN|} \quad R_x = \frac{VUP \times R_z}{|VUP \times R_z|} \quad R_y = R_z \times R_x$$

$$R = \begin{bmatrix} R_{1x} & R_{2x} & R_{3x} & 0 \\ R_{1y} & R_{2y} & R_{3y} & 0 \\ R_{1z} & R_{2z} & R_{3z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Perspective Projections

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Step3:

$$\begin{bmatrix} 1 & 0 & 0 & prp_u \\ 0 & 1 & 0 & prp_v \\ 0 & 0 & 1 & prp_n \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Step4:

Perspective Projections

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$$\begin{bmatrix} 1 & 0 & shear_x & 0 \\ 0 & 1 & shear_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$shear_x = -\frac{dop_x}{dop_z}$$

$$shear_y = -\frac{dop_y}{dop_z}$$

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Step5:

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$$VRP' = S_{step4} \cdot T_{step3} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{2vrp'_z}{(u_{\max}-u_{\min})(vrp'_z+B)} & 0 & 0 & 0 \\ 0 & \frac{2vrp'_z}{(v_{\max}-v_{\min})(vrp'_z+B)} & 0 & 0 \\ 0 & 0 & \frac{-1}{vrp'_z-B} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Perspective Projections

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$$\left[\begin{array}{l} \text{Orthographic} \\ \text{Projection} \\ \text{Matrix} \end{array} \right] \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -D \\ 0 & 0 & \left(\frac{1}{D} \right) & 0 \end{array} \right] \quad \text{Where : } D = \cot\left(\frac{fov}{2}\right)$$

$$\left[\begin{array}{c} x \\ y \\ (z - D) \\ \left(\frac{z}{D} \right) \end{array} \right] \Rightarrow \left[\begin{array}{ccc} D \frac{x}{z} & D \frac{y}{z} & D \frac{z - D}{z} \end{array} \right] = [X \quad Y \quad Z]$$