Requirements

- 2 hours
- Closed book. You may bring one side of a page of notes.
- Calculators are permitted.
- Write your answers in a blue book or on separate sheets of paper. Do not write them on the exam questions and do not turn in the exam questions.
- You may use without rederivation the results derived in class, the book, the homework, handouts, or recitation.
- Write and sign the honor code pledge on your exam. ("I have neither given nor received aid on this exam, nor have I concealed any honor code violations.")
- Estimated points for each part of each problem are marked in square brackets. While grading, I sometimes find that changes are needed.

Testmanship:

- Problem 1 contains only short answer questions. No explanations or justifications are required.
- In some problems you are asked to "justify your answer", in which case you will be graded on the logic of your argument. Be brief and precise.
- In other problems, you are not required to show your work or your approach. However, if you do show your work and approach, you can receive partial credit for a partially correct answer or approach. So it is recommended that you always show your work and justify your approach (briefly).
- Don't try to cram too much on a page. Give yourself room to work and to modify your answers.
- If you use a blue book, start each problem on a new "left" page. This gives you room to work, to make changes to previous problems, and to see (and check) as much as possible of your solution at one time.
- If you don't use a blue book, start each problem on a new piece of paper and write on only one side of the page. This gives you room to make changes to previous problems and enables you to see (and check) as much as possible of your solution at one time. (You cannot see both sides of a page at once.)
1. Short answer questions. Be brief and precise. No explanations or elaborations are needed.

(a) **State** the definition of a "countably infinite set"?

(b) **State** the definition of a "σ-algebra"?

(c) Let \( \Omega = [3,7] \). **Give** a brief precise definition of the "Borel σ-algebra for \( \Omega \)". (You need not define "σ-algebra", but you must define any terms in your definition that are not elementary set terminology.)

(d) Let \( X \) be a random variable. **Define** what it means for \( X \) to be a "discrete random variable". That is, state the defining characteristic. (You do not have to define "random variable".)

(e) **State** the "fundamental theorem of expectation", also called the "law of the unconscious statistician"?

2. When the sample space of a probability model is \( \Omega = [0,1] \), why don't we take the event space to be the power set of \( \Omega \)?

3. Consider two σ-algebras on the sample space \( \Omega = \{a,b,c,d\} \):

\[ \mathcal{E}_1 = \{ \emptyset, \{a\}, \{b,c,d\}, \{a,b,c,d\} \} \]

\[ \mathcal{E}_2 = \{ \emptyset, \{a,b\}, \{c,d\}, \{a,b,c,d\} \} \]

(a) Is \( \mathcal{E}_1 \cap \mathcal{E}_2 \) a σ-algebra?

(b) Is \( \mathcal{E}_1 \cup \mathcal{E}_2 \) a σ-algebra? **Justify your answer.**

4. Suppose \( A \) and \( B \) are events such that

\[ P(A \cup B) = P(A) + P(B) - .1 \quad \text{and} \quad P(A|B) = .2 \]

**Find** the probabilities of the all of the events listed below for which the above information is sufficient.

(a) \( P(A^c|B) \)

(b) \( P(A) \)

(c) \( P(B) \)

(d) \( P(A \cup B \cup (A^c \cap B^c)) \)
5. Given a probability model \((\Omega, \mathcal{E}, \mathbb{P})\) with \(\Omega = [0,1]\), \(\mathcal{E} = \text{Borel } \sigma\)-algebra for \(\Omega\), and probability measure \(\mathbb{P}\) such that for any open subinterval \((a,b)\) in \(\Omega\)

\[
\mathbb{P}((a,b)) = \begin{cases} 
\frac{1}{2} + \frac{b-a}{2}, & \text{if } a < \frac{1}{2} < b \\
\frac{b-a}{2}, & \text{else}
\end{cases}
\]

Find the probability of the interval \([\frac{1}{2}, \frac{3}{4}]\). Justify your answer.

6. Three cooks, A, B, C bake a special kind of cake, and with respective probabilities .02, .03, and .05 the cakes they bake fail to rise. In the restaurant where they work, A bakes 50% of the cakes, B 30% and C 20%. If a cake fails to rise, what is the probability that it was baked by A?

7. The Bet-Your-Life Casino charges \(P\) dollars to play a game in which the customer tosses a coin until heads appears. If this happens on the \(k\)th toss, the customer wins \(d^k\) dollars. The coin is biased in such a way that heads occurs with probability \(q\). What should be the value of \(d\) so that on the average the customer wins \(P\) dollars. Express your answer in terms of \(P\) and \(q\). Simplify your answer as much as possible.

Extra Credit

8. Name another reference that mentions the "law of the unconscious statistician".

[120 points total]