Notes on Mean Ergodicity

Definition: A wide sense stationary random discrete-time process $\{X_k\}$ is *mean ergodic* if

$$E\left(\frac{1}{n}\sum_{k=1}^{n} X_k - EX_1\right)^2 \rightarrow 0 \text{ as } n \rightarrow \infty \tag{1}$$

Equivalently, $\{X_k\}$ is mean ergodic if $\text{var}(T_n) \rightarrow 0$ as $n \rightarrow \infty$, where T_n is the sample average of $X_1,\ldots,X_n$; i.e.

$$T_n = \frac{1}{n}\sum_{k=1}^{n} X_k$$

Theorem: A wide-sense stationary random discrete-time process $\{X_k\}$ is *mean ergodic* if and only if its covariance function $K_X(k)$ satisfies

$$\frac{1}{n}\sum_{k=1}^{n} K_X(k) \rightarrow 0 \quad \text{as } n \rightarrow \infty \tag{2}$$

Corollary: Either of the following is sufficient (but not necessary) for mean ergodicity:
(i) $K_X(k) \rightarrow 0$ as $n \rightarrow \infty$, or (ii) $\sum_{k=1}^{\infty} K_X(k) < \infty$

Note: A similar Theorem and Corollary hold in the continuous-time case.

Proof of Theorem:

We first show that $\{X_k\}$ is mean ergodic iff

$$\frac{1}{n}\sum_{k=1}^{n} \left(1 - \frac{k}{n}\right) K_X(k) \rightarrow 0 \quad \text{as } n \rightarrow \infty; \tag{3}$$

i.e. (1) $\Leftrightarrow$ (3). We will subsequently complete the proof by showing (1) $\Rightarrow$ (2) $\Rightarrow$ (3) .

Proof that (1) $\Leftrightarrow$ (3):

$$\text{var}(T_n) = \text{var}\left(\frac{1}{n}\sum_{k=1}^{n} X_k\right) = \frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n} \text{cov}(X_j X_k)$$

$$= \frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n} K_X(j-k) = \frac{1}{n^2}\sum_{k=1}^{n-1} 2(n-k)\, K_X(k) + \frac{1}{n^2} nK_X(0) \tag{4}$$

where this follows from the fact that both of the last two expressions can be seen to equal the sum of all elements in the matrix (it's a covariance matrix) shown below

$$\begin{matrix} K_X(0) & K_X(1) & K_X(2) & \ldots\ldots & K_X(n-1) \\ K_X(1) & K_X(0) & K_X(1) & \ldots\ldots & K_X(n-1) \\ K_X(2) & K_X(1) & K_X(0) & \ldots\ldots & K_X(n-1) \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ K_X(n) & K_X(n-1) & & \ldots\ldots & K_X(0) \end{matrix}$$

Since the second term in (4) goes to zero, we see that $\text{var}(T_n) \rightarrow 0$ iff (3) holds. In other words, (1) $\Leftrightarrow$ (3) .

Proof that (1) $\Rightarrow$ (2):

Let us assume that (1) holds. Observe that

$$\begin{aligned}
\mathrm{cov}(X_n,T_n) &= E\,(X_n-EX)(T_n-EX) = E\,\frac{1}{n}\sum_{k=1}^{n}(X_n-EX)(X_k-EX) \\
&= \frac{1}{n}\sum_{k=1}^{n}\mathrm{cov}(X_n,X_k) = \frac{1}{n}\sum_{k=1}^{n}K_X(n-k) \\
&= \frac{1}{n}\sum_{k'=0}^{n-1}K_X(k') \qquad \text{(letting } k' = n-k) \\
&= \frac{1}{n}\sum_{k=1}^{n}K_X(k) + \frac{1}{n}(K_X(0)-K_X(n))
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n}K_X(k) &= \mathrm{cov}(X_n,T_n) - \frac{1}{n}(K_X(0)-K_X(n)) \\
&\le \sqrt{\mathrm{var}(X_n)\mathrm{var}(T_n)} - \frac{1}{n}(K_X(0)-K_X(n))
\end{aligned}$$

Now (1) implies that $\mathrm{var}(T_n) \to 0$ as $n\to\infty$. Since the second term in the above also goes to zero, we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}K_X(k) = 0$$

which is (2).

Proof that (2) $\Rightarrow$ (3):

First we note that

$$\sum_{k=1}^{n}(n-k)\; K_X(k) = \sum_{j=1}^{n-1}\sum_{k=1}^{j}K_X(k) \tag{5}$$

which follows from the fact that both sides represent the sum of all terms in the following array

$$\begin{array}{llllll}
K_X(1) & K_X(1) & \ldots\ldots & K_X(1) & K_X(1) & \text{(n-1 terms)} \\
K_X(2) & K_X(2) & \ldots\ldots & K_X(2) & & \text{(n-2) terms)} \\
\cdot & & & & & \\
\cdot & & & & & \\
\cdot & & & & & \\
K_X(n-2) & K_X(n-2) & & & & \text{(2 terms)} \\
K_X(n-1) & & & & & \text{(1 term)}
\end{array}$$

(The right hand side is the sum by rows.). Let us now fix a small number $\varepsilon > 0$. Assuming (2), the definition of a limit implies there exists n_o such that

$$\left|\frac{1}{n}\sum_{k=1}^{n}K_X(k)\right| \le \varepsilon\,, \quad \text{for all } n \ge n_o\,. \tag{6}$$

Then for any $n \geq n_o$,

$$\left| \frac{1}{n} \sum_{k=1}^{n} (1 - \frac{k}{n}) K_X(k) \right| = \left| \frac{1}{n_2} \sum_{k=1}^{n} (n-k) K_X(k) \right|$$

$$= \left| \frac{1}{n^2} \sum_{j=1}^{n-1} \sum_{k=1}^{j} K_X(k) \right| \qquad \text{(using (5))}$$

$$\leq \frac{1}{n^2} \sum_{j=1}^{n-1} \left| \sum_{k=1}^{j} K_X(k) \right| \qquad \text{(abs. value of sum} \leq \text{sum of abs. val's)}$$

$$= \frac{1}{n^2} \sum_{j=1}^{n_o} \left| \sum_{k=1}^{j} K_X(k) \right| + \frac{1}{n^2} \sum_{j=n_o+1}^{n-1} j \left| \frac{1}{j} \sum_{k=1}^{j} K_X(k) \right| \qquad \text{(sum divided in two)}$$

$$\leq \frac{1}{n^2} \sum_{j=1}^{n_o} j\, K_X(0) + \frac{1}{n^2} \sum_{j=n_o+1}^{n-1} j\, \varepsilon \qquad \text{(using } K_X(k) \leq K_X(0) \text{ and (6))}$$

$$\leq \frac{1}{n^2} n_o^2 K_X(0) + \varepsilon$$

$$\leq 2\varepsilon \quad \text{when } n \text{ is large} \qquad \text{(since the first term goes to zero)}$$

At this point we have shown that for any $\varepsilon > 0$,

$$\left| \frac{1}{n} \sum_{k=1}^{n} (1 - \frac{k}{n}) K_X(k) \right| \leq 2\varepsilon \text{ when } n \text{ is sufficiently large.}$$

By the definition of a limit, this means that

$$\frac{1}{n} \sum_{k=1}^{n} (1 - \frac{k}{n}) K_X(k) \to 0 \quad \text{as } n \to \infty$$

and completes the proof that $(2) \Rightarrow (3)$.