Requirements

- 2 hours
- Closed book. You may bring one side of a page (8.5”×11”) of notes.
- Calculators are NOT permitted.
- Write your answers in a blue book or on separate sheets of paper. (Bring your own.) Do not write them on the exam questions and do not turn in the exam questions. If you use separate sheets of paper, write on only one side of each sheet.
- Submit your answers in the order of the questions.
- You may use without rederivation the results derived in class, the book, the homework, handouts, or recitation.
- Write and sign the honor code pledge on your exam. (“I have neither given nor received aid on this exam, nor have I concealed any honor code violations.”)
- Estimated points for each part of each problem are marked in square brackets. While grading, I sometimes find that changes are needed.

Testmanship:

- Problem 1 contains only short answer questions. No explanations or justifications are required.
- In some problems you may be asked to "justify your answer" or to "prove" or "show" something, in which case you will be graded on the logic of your argument. Be brief and precise.
- In other problems, you are not required to show your work or your approach. However, if you do show your work and approach, you can receive partial credit for a partially correct answer or approach. So it is recommended that you always show your work and justify your approach (briefly). This is especially important in problems with multiple parts. If you make a mistake in an earlier part, you can still receive much or full credit for a later part, provided you clearly indicate your approach in the later parts. For example, in a later part, you can write the formulas you substitute into before substituting answers from earlier parts.
- Do not try to cram too much on a page. Give yourself room to work and to modify your answers.
- If you use a blue book, start each problem on a new "left" page. This gives you room to work, to make changes to previous problems, and to see (and check) as much as possible of your solution at one time.
- If you do not use a blue book, start each problem on a new piece of paper and write on only one side of the page. This gives you room to make changes to previous problems and enables you to see (and check) as much as possible of your solution at one time. (You cannot see both sides of a page at once.)
1. Short answer questions. No explanation or derivations needed.

(a) State the definition of the Cartesian product of two events \( A \) and \( B \).

(b) State the definition of the power set of a set \( A \).

(c) State the definition of the Borel probability measure for the sample space \( \Omega = [0,1] \).

(d) State what it means for a compound experiment \( X=(X_1,X_2) \) to have independent components. (If your statement uses the word “independent”, make sure you specify what this means.)

(e) State Bayes rule for conditional probability.

2. Let \( \Omega = [0,10] \), let \( \mathcal{E} \) be the Borel \( \sigma \)-algebra for \( \Omega \), and let \( P \) be the Borel probability measure on \( \mathcal{E} \).

(a) Express the following set as an interval.

\[
F = \bigcap_{n=1}^{\infty} \left( 3-\frac{1}{n^2}, 5+\frac{2}{n} \right) \cup \bigcup_{n=2}^{\infty} \left( 4+\frac{1}{n^2}, 6-\frac{2}{n} \right)
\]

(b) Find \( P(F) \).

3. In a probability model \( (\Omega,\mathcal{E},P,X) \), \( A, B \) and \( C \) are events in \( \mathcal{E} \) with nonzero probabilities such that \( A \subset C \), \( B \) and \( C \) are disjoint, and \( A \cup B \) is independent of \( C \). Derive an expression for the probability of \( C \) in terms of the probabilities of \( A \) and \( B \). Simplify as much as possible. “Derive” means you must show and justify the steps that lead to your solution (briefly).

4.

(a) Find a probability model for the following experiment.

A (fair) coin is tossed, producing a Heads or a Tails, and a (fair) wheel is spun, which stops at some angle in the interval \( [0, 2\pi] \). When the coin is Heads, the outcome of the experiment is the angle at which the wheel stops. When the coin is Tails, the outcome of the experiment is twice the angle.

(b) Find the probability that the outcome of the experiment is between 0 and 1.
5. Consider the sample space $\Omega = \{a, ab, abc, abcd\}$.

(a) **Find** the event space $E = \sigma(\{\{a, ab\}, \{ab, abc\}\})$, i.e. list its elements or precisely specify $E$ in some other way.

(b) Suppose $P$ is a probability measure on $E$ such that $P(\{a, ab\}) = 0.5$ and $P(\{ab, abc\}) = 0.7$. Is it possible to determine $P(\{ab\})$? If yes, **find** $P(\{ab\})$ and **justify** your answer. If not, **find** two different probability measures on $E$ such that each assigns probability 0.5 to $\{a, ab\}$ and 0.7 to $\{ab, abc\}$ but gives a different value to $P(\{ab\})$.

6. Professor Random has taught probability for many years. She has found that 80% of the students who do the homework pass the exam, while 10% of students who don’t do the homework pass the exam. In her present class 60% of the students do the homework. Suppose a student is randomly chosen from the class.

(a) **What is** the probability that the student has passed the exam?

(b) If the student has failed the exam, **what is** the probability that he/she does the homework?

7. Ten percent of parts produced by a certain factory production line are defective. Parts are defective or not defective, independently of other parts. **Find** an expression for the probability that more than four parts are defective in a batch of 15? **Justify** your answer (briefly).

[125 points total]