Requirements

- 2 hours
- Closed book. You may bring two sides of a page (8.5"×11") of notes.
- Calculators are NOT permitted.
- Write your answers in a blue book or on separate sheets of paper. (Bring your own.) Do not write them on the exam questions and do not turn in the exam questions. If you use separate sheets of paper, write on only one side of each sheet.
- Submit your answers in the order of the questions.
- You may use without rederivation the results derived in class, the book, the homework, handouts, or recitation.
- Write and sign the honor code pledge on the front of your exam. ("I have neither given nor received aid on this exam, nor have I concealed any honor code violations.")
- Estimated points for each part of each problem are marked in square brackets. While grading, I sometimes find that changes are needed.

Testmanship:

- Problem 1 contains only short answer questions. No explanations or justifications are required.
- In some problems you may be asked to "justify your answer" or to "prove" or "show" something, in which case you will be graded on the logic of your argument. Be brief and precise.
- In other problems, you are not required to show your work or your approach. However, if you do show your work and approach, you can receive partial credit for a partially correct answer or approach. So it is recommended that you always show your work and justify your approach (briefly). This is especially important in problems with multiple parts. If you make a mistake in an earlier part, you can still receive much or full credit for a later part, provided you clearly indicate your approach in the later parts. For example, in a later part, you can write the formulas you substitute into before substituting answers from earlier parts.
- Do not try to cram too much on a page. Give yourself room to work and to modify your answers.
- If you use a blue book, start each problem on a new "left" page. This gives you room to work, to make changes to previous problems, and to see (and check) as much as possible of your solution at one time.
- If you do not use a blue book, start each problem on a new piece of paper and write on only one side of the page. This gives you room to make changes to previous problems and enables you to see (and check) as much as possible of your solution at one time. (You cannot see both sides of a page at once.)
1. Short answer questions. No explanations or derivations needed. **However, be sure to define terms that you use.**

   (a) **Define** what it means for $X$ and $Y$ to be identical random variables. For full credit, your definition should apply to all types of random variables.

   (b) **Define** the correlation coefficient for random variables $X$ and $Y$. For full credit, your definition should apply to all types of random variables.

   (c) **Define** what it means for random variables $X$ and $Y$ to be independent. For full credit, your definition should apply to all types of random variables.

   (d) For a discrete random variable, **state** the fundamental theorem of expectation, which is also called LOTUS, the law of the unconscious statistician.

   (e) **State** the definition of an “absolutely continuous random variable”. You need not define “random variable”.

2. Random variables $X$ and $Y$ are modelled by the functions shown below. The underlying probability model is $(\Omega_u,\mathcal{E}_u,P_u,U)$ where $\Omega_u = [0,1]$, $\mathcal{E}_u$ is the Borel $\sigma$-algebra on $[0,1]$, and $P_u$ is the Borel probability measure.

   \[
   X(\omega) = \begin{cases} 
   0, & 0 \leq \omega < 1/2 \\
   1, & 1/2 \leq \omega \leq 1 
   \end{cases} \quad Y(\omega) = \begin{cases} 
   \omega, & 0 \leq \omega \leq 1/2 \\
   1, & 1/2 < \omega \leq 1 
   \end{cases}
   \]

   These functions are measurable.

   (a) **What** type of random variable is $X$? **Justify** your answer.

   (b) **What** type of random variable is $Y$? **Justify** your answer.

   (c) **Find** $\Pr(X<Y)$. **Justify** your answer.

   (d) **Are** $X$ and $Y$ independent? **Justify** your answer.

3. The number of customers who enter the Wolverine Ice Cream Shop between 1 and 2 PM is a Poisson random variable $X$ with parameter $\lambda$. The first customer to enter in that time period writes “1” on a piece of paper and puts it in a box at the back of the store. The second customer to enter in this time period writes “2” on a piece of paper and puts it in the box. All subsequent customers do the same thing. That is, the $n$th customer in this time period writes “$n$” on a piece of paper and puts it in the box. At 2 PM the owner writes $X+1$ on a piece of paper and puts it in the box. Finally, one piece of paper is drawn at random from the box. If the paper has the number $n$ written on it and $n \leq X$, then the $n$th customer wins an ice cream pie.

   If the second customer to enter the shop wins the pie, **what is** the probability that exactly four customers entered the shop between 1 and 2 PM? **Give** as simple an expression for the answer as possible.

   (20)
4. There are 29 stocks on the Get Rich Quick Stock Exchange. The price of each stock (in whole dollars) is geometric(p). (p is the same for all stocks.) Prices of different stocks are independent. Find an expression for the probability that at least one stock costs more than 10 dollars. Simplify as much as possible.

5. X and Y are independent nonnegative random variables, and Z = X/Y

We are given the following pieces of information:

\[ \mathbb{E}[X^2] = 16 \]

Y is a discrete random variable with \( p_Y(1) = \frac{1}{2}, \ p_Y(3) = \frac{1}{2} \)

\[ \Pr(Z \geq 3) = .5 \]

(a) Find an expression for \( \mathbb{E}[Z] \).

(b) Find the best possible lower bound to \( \mathbb{E}[X] \). That is, find a number \( c \) such that

\[ \mathbb{E}[X] \geq c \]

Full credit goes to the largest \( c \) that can be found from the information given in this problem. Justify your answer, i.e. show the steps of your work and justify the steps as needed.

(c) Find the best possible upper bound to \( \mathbb{E}[X] \). That is, find a number \( d \) such that

\[ \mathbb{E}[X] \leq d \]

Full credit goes to the smallest \( d \) that can be found from the information given in this problem. Justify your answer, i.e. show the steps of your work and justify the steps as needed.

6. X and Y are independent random variables and \( Z = X + Y \). X is an exponential random variable with mean 1. The characteristic function of \( Z \) is

\[ \phi_Z(v) = \frac{1}{4(1-jv)} + \frac{3}{4(1-jv)} e^{j2v} \]

Find the probability distribution of Y. (It is up to you to decide how to describe the probability distribution.)