1. Gubner, Prob. 22 (a), (d), p. 50.

2. Gubner, Prob. 23, p. 50.


5. For each of the following cases, determine if \( f \) is a valid function with domain \( A \) and range \( B \). For those that are valid functions, determine if they are one-to-one, onto, continuous, monotonic (if so, state the type of monotonicity), and find the inverse image of the set \((-1, 2)\)
   
   (a) \( A = [0, 1], \ B = [-1, 1], \ f(x) = \{ y \in B : y^2 = x \}, \) for all \( x \in A \)
   
   (b) \( A = [-1, 1], \ B = [-\pi, \pi], \ f(x) = \{ y \in B : \sin y = x \}, \) for all \( x \in A \)
   
   (c) \( A = [0, 1], \ B = [-1, -1], \ f(x) = \begin{cases} 1, & \text{if } x \in [\frac{1}{4}, 1] \\ 0, & \text{else} \end{cases}, \) for all \( x \in A \)

6. Let \( \Omega = \{1, 2, 3, 4\} \). Which, if any, of the following event spaces are \( \sigma \)-algebras?
   
   (a) \( \mathcal{E}_a = \{ \emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\} \} \)
   
   (b) \( \mathcal{E}_b = \{ \emptyset, \{1, 2\}, \{1, 2, 3, 4\} \} \)
   
   (c) \( \mathcal{E}_c = \{ \emptyset, \{1, 2, 3, 4\} \} \)
   
   (d) \( \mathcal{E}_d = \text{power set of} \ \Omega. \)
   
   (e) Of those event spaces that are \( \sigma \)-algebras, do any contain any of the others that are \( \sigma \)-algebras? If so, list which contain which.

7. Let \( \Omega \) be a sample space, and let \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) be two \( \sigma \)-algebras of subsets of \( \Omega \). Show that \( \mathcal{E} = \mathcal{E}_1 \cap \mathcal{E}_2 \) is a \( \sigma \)-algebra.

8. Let \( \Omega = \{1, 2, 3, \ldots\} \) be a sample space. Let \( \mathcal{E} = \text{collection of all finite subsets of} \ \Omega. \) Is \( \mathcal{E} \) a \( \sigma \)-algebra? Prove your answer. (To prove that \( \mathcal{E} \) is a \( \sigma \)-algebra you must show that it satisfies the conditions that define a \( \sigma \)-algebra. To disprove it, you need only give an example of something that violates one of the conditions that a \( \sigma \)-algebra must satisfy.)

9. Let \( \Omega = [0, 1] \) be a sample space and let \( \mathcal{B} = \{ [0, \frac{1}{2}], [\frac{1}{2}, 1] \} \) denote a collection of two events. Find the \( \sigma \)-algebra generated by \( \mathcal{B} \).
10. Let $\Omega = [0,1]$, let $B$ be the Borel $\sigma$-algebra, and let $P$ be the unique probability measure on $B$ such that $P([a,b]) = b - a$ for all intervals $[a,b]$, $0 \leq a < b \leq 1$. Show that each of the following sets is in $B$ and finds its probability. (Start from the definition of the Borel $\sigma$-algebra.)

(a) \{1/2\}
(b) \{1, \frac{1}{2}, \frac{1}{4}, \ldots\}

11. Let $\Omega = [0,1]$ and $E$ = the power set of $\Omega$. Which of the following set functions on $E$ are valid probability measures? Prove your answers. (To prove one is valid, you must show it satisfies the axioms. To disprove the validity of one, you need only given an example of something that violates an axiom or a property that derives from an axiom.)

(a) $P(A) = \begin{cases} 1, & \text{if } 1/4 \in A \\ 0, & \text{if not} \end{cases}$

(b) $P(A) = \begin{cases} 1, & \text{if } [0,1/4] \subset A \\ 0, & \text{if not} \end{cases}$

12. In an imaginary two-dimensional "world", a dart is thrown horizontally from the right into the "tube" shown below. The dart sticks wherever it hits the circular arc. If it enters the tube at "height" $h$ from the bottom of the tube, then it sticks to the arc at the point on the arc that is distance $h$ from the bottom.

Suppose we are told that the height $h$ of the dart (from the bottom) is random and equally likely to be anywhere between 0 and 2 inches. As illustrated above, let $\theta$ denote the angle corresponding to the point at which the dart hits the arc, $0 \leq \theta \leq \pi$.

Find a probability model for the recorded angle $\theta$. You need not be concerned with the event space. Just say that the event space is some $\sigma$-algebra that contains all subintervals of [0, $\pi$]. For your probability measure, find a function $f$ for which you can show that

$$P([a,b]) = \int_a^b f(\theta) \, d\theta$$

for any closed interval $[a,b]$ in the sample space.

(Your integral formula will also apply to open or half open intervals.)