1. Consider the discrete-time binomial random process $Y_n = \sum_{i=1}^{n} X_i$, $n = 1, 2, 3, ...$ where $X_n$ is a Bernoulli random process with $p_X(1) = q$, $p_X(0) = 1-q$. Show that the autocorrelation function is $$R_X(k,m) = kq(1+(m-1)q), \text{ when } m \geq k$$

2. Consider the discrete-time Gaussian autoregressive random process $Y_n$, which is defined by the property that

$$Y_n = a Y_{n-1} + X_n, \quad n = 1, 2, ...$$

where $X_n$ is an IID Gaussian random process with mean $0$ and variance $\sigma^2_X$, and where $X_n$ is uncorrelated with $Y_{n-1}$, $Y_{n-2}$, ..., $Y_0$. Assume that $Y_n$ is widesense stationary. Assume also that $Y_0$ is Gaussian.

(a) Find the mean function $\mu_Y(n)$. (Hint: Use the fact that the mean of $Y_n$ and $X_n$ do not change with $n$.)

(b) Find the variance of $Y_n$ in terms of $\sigma^2_X$ and $a$. (Hint: square and take the expected values of both sides of $Y_n = a Y_{n-1} + X_n$ and use the widesense stationarity.)

(c) Show that the autocorrelation function is $R_Y(k) = a |k| \frac{1}{1-a^2} \sigma^2_X$. ($R_X(k) = R_X(n, n+k)$ for any $n$)

(d) Find the joint probability density of $X_1, X_3$.

(e) If $a = .9$, find $Pr(X_3 > X_1 + 1)$.

3. Consider the random process $X_t = a \cos(\omega t + \Theta)$, $t \in T = (-\infty, \infty)$, where $a$ and $\omega$ are constants, and $\Theta$ is a random variable uniformly distributed between $0$ and $2\pi$.

(a) Find the density of $X_t$.

(b) Find the mean function of $X_t$

(c) Find the autocorrelation function of $X_t$.

(d) Is $X_t$ widesense stationary?

(e) Does $\frac{1}{T} \int_{-T}^{T} X_t \, dt \to EX_t$ with probability 1? If it does, then one can say it is “mean ergodic” or has “ergodicity of the mean”.

4. Consider a random process $X_t$ described by the following functional model. The underlying experiment is $(\Omega_u, E_u, P_u, U)$ where $\Omega_u = \{1,2,3,4\}$, $E_u$ = power set of $\Omega_u$, $P_u(A) = |A|/4$ for any $A \in E_u$. The index set is $T = (-\infty, \infty)$. The function $X_t(\omega)$ is

$$X_t(1) = 1, \quad X_t(2) = -2, \quad X_t(3) = \sin \pi t, \quad X_t(4) = \cos \pi t.$$ 

(a) Sketch the sample functions of this random process.

(b) Find the mean function.

(c) Find the autocorrelation function.

(d) Is the process wide-sense stationary?

(e) Is the process stationary in the strict sense?

(f) Find $Pr(-.5 < X_.5 < 2.5, -.5 < X_1 < .5)$. (Hint: It helps to draw pictures.)

(g) Find $Pr(-.8 < X_{-.25} < .8, -.8 < X_{.25} < .8)$

(h) Find $Pr(0 \leq X_t \leq 1, 0 < t < 1/2)$

Continued on next page.
5. Suppose $X_t$ is a wide-sense stationary random process with index set $T = (-\infty, \infty)$ mean zero and autocorrelation function $R_X(\tau)$. Consider the following random process with the same index set

$$Y_t = \int_{t-a}^{t} X_s \, ds,$$

(a) Find the mean function of $Y_t$.
(b) Find the autocorrelation function of $Y_t$.
(c) Is $Y_t$ wide-sense stationary?

6. Let $X_t, \ t \in T = (-\infty, \infty)$, be the random process defined by

$$X_t = A \cos(\omega t) + B \sin(\omega t)$$

where $\omega$ is a constant and $A$ and $B$ are IID random variables with mean $0$, variance $\sigma^2$, and $E[A^3] \neq 0$.

(a) Show that $X_t$ is WSS.
(b) Show that $X_t$ is not stationary in the strict sense. Hint: Consider $E[X_t^3]$.

7. Let $X_t$ be a stationary, ergodic continuous-time random process and let $Y_t = (X_t)^2$.

(a) Is $Y_t$ stationary?
(b) Is $Y_t$ ergodic?

8. Let $X_n$ be a Bernoulli process with $p_X(1) = q = 1 - p_X(0)$. Let $\{Y_t: -\infty < t < \infty\}$ be the continuous-time random process defined by

$$Y_t = X_n \quad \text{if} \quad n \leq t < n+1$$

(a) For $s < t$ find the joint pmf of $Y_t, Y_s$: $p_{Y_t,Y_s}(y_1,y_2)$
(b) Find the autocorrelation function of $Y_t$.
(c) Is $\{Y_t\}$ wide stationary? Stationary?
(d) Is it mean ergodic?