1. Consider the discrete-time binomial random process \( Y_n = \sum_{i=1}^{n} X_i, \ n = 1,2,3,... \) where \( X_n \) is a Bernoulli random process with \( p_X(1) = q, \ p_X(0) = 1-q \). Show that its autocorrelation function is 
\[ R_Y(k,m) = kq( 1+(m-1)q), \] when \( m \geq k \)

2. Consider the discrete-time Gaussian autoregressive random process \( Y_n \), which is defined by the property that 
\[ Y_n = a Y_{n-1} + X_n, \ n = 1,2,... \] 
where \( X_n \) is an IID Gaussian random process with mean 0 and variance \( \sigma_X^2 \), and where \( X_n \) is uncorrelated with \( Y_{n-1}, Y_{n-2}, ... , Y_0 \). Assume that \( Y_n \) is wide-sense stationary. Assume also that \( Y_0 \) is Gaussian with zero mean.
(a) Find the mean function \( \mu_Y(n) \). (Hint: Use the fact that the mean of \( Y_n \) and \( X_n \) do not change with \( n \).)
(b) Find the variance of \( Y_n \) in terms of \( \sigma_X^2 \) and \( a \). (Hint: square and take the expected values of both sides of \( Y_n = a Y_{n-1} + X_n \) and use the wide-sense stationarity.)
(c) Show that the autocorrelation function is 
\[ R_Y(k) = a |k| \frac{1}{1-a^2} \sigma_X^2. \] (\( R_Y(k) \triangleq R_Y(n,n+k) \) for any \( n \))
(Hint: Show that \( R_Y(k) = a R_Y(k-1) \))
(d) Find the joint probability density of \( Y_1, Y_3 \)
(e) If \( a = .9 \), find \( \Pr(Y_3 > Y_1 + 1) \).

3. Consider the random process \( X_t = a \cos(\omega t + \Theta) \), \( t \in T = (-\infty, \infty) \), where \( a \) and \( \omega \) are constants, and \( \Theta \) is a random variable uniformly distributed between 0 and \( 2\pi \).
(a) Find the density of \( X_t \).
(b) Find the mean function of \( X_t \)
(c) Find the autocorrelation function of \( X_t \).
(d) Is \( X_t \) wide-sense stationary?
(e) Does \( \frac{1}{T} \int_{T} X_t \, dt \to E X_t \) with probability 1? If it does, then one can say it is “mean ergodic” or has “ergodicity of the mean”.

4. Consider a random process \( X_t \) described by the following functional model. The underlying experiment is \( (\Omega_u, E_u, P_u, U) \) where \( \Omega_u = \{1,2,3,4\} \), \( E_u = \) power set of \( \Omega_u \), \( P_u(A) = |A|/4 \) for any \( A \in E_u \). The index set is \( T = (-\infty, \infty) \). The function \( X_t(\omega) \) is
\[ X_t(1) = 1, \ X_t(2) = -2, \ X_t(3) = \sin \pi t, \ X_t(4) = \cos \pi t. \]
(a) Sketch the sample functions of this random process.
(b) Find the mean function.
(c) Find the autocorrelation function.
(d) Is the process wide-sense stationary?
(e) Is the process stationary in the strict sense?
(f) Find \( \Pr(-2.5 < X_{.5} < 2.5, -.5 < X_1 < .5) \). (Hint: It helps to draw pictures.)
(g) Find \( \Pr(-.8 < X_{.25} < .8, -.8 < X_{.25} < .8) \)
(h) Find \( \Pr(0 \leq X_t \leq 1, 0 < t < 1/2) \)

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5. Suppose $X_t$ is a wide-sense stationary random process with index set $T = (-\infty, \infty)$ mean zero and autocorrelation function $R_X(\tau)$. Consider the following random process with the same index set

$$Y_t = \int_{t-a}^{t} X_s \, ds,$$

(a) Find the mean function of $Y_t$.
(b) Find the autocorrelation function of $Y_t$.
(c) Is $Y_t$ wide-sense stationary?

6. Let $X_t$, $t \in T = (-\infty, \infty)$, be the random process defined by

$$X_t = A \cos(\omega t) + B \sin(\omega t)$$

where $\omega$ is a constant and $A$ and $B$ are IID random variables with mean 0, variance $\sigma^2$, and $EA^3 \neq 0$.
(a) Show that $X_t$ is WSS.
(b) Show that $X_t$ is not stationary in the strict sense. Hint: Consider $E X_t^3$.

7. Let $X_t$ be a stationary, ergodic continuous-time random process and let $Y_t = (X_t)^2$.
(a) Is $Y_t$ stationary?
(b) Is $Y_t$ ergodic?

8. Let $X_n$ be a Bernoulli process with $p_X(1) = q = 1 - p_X(0)$. Let $\{Y_t: -\infty < t < \infty\}$ be the continuous-time random process defined by

$$Y_t = X_n \quad \text{if} \quad n \leq t < n+1$$

(a) For $s < t$ find the joint pmf of $Y_t, Y_s$: $p_{Y_t, Y_s}(y_1, y_2)$
(b) Find the autocorrelation function of $Y_t$.
(c) Is $\{Y_t\}$ wide stationary? Stationary?
(d) Is it mean ergodic?