Recitation 5 (Review)

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Topics:

1. Sets
   (a) Cartesian product: \( A \times B = \{(x, y) : x \in A, y \in B\} \neq A \cap B \)
   (b) DeMorgan’s Law: \((A \cup B)^C = (A^C \cap B^C), (A \cap B)^C = (A^C \cup B^C)\)
   (c) \([a, b) = \lim_{n \to \infty}[a, b - \frac{1}{n}] = \bigcup_{n=1}^{\infty}[a, b - \frac{1}{n}]\)
   (d) \([a, b] = \lim_{n \to \infty}[a, b + \frac{1}{n}] = \bigcap_{n=1}^{\infty}[a, b + \frac{1}{n}]\)

2. \(\sigma\)-algebra \(E\)
   Axioms: S1: \(\phi \in E\)
   S2: If \(A \in E\), then \(A^C \in E\)
   S3a: If \(A, B \in E\), then \(A \cup B \in E\)
   S3b: If \(A_1, A_2, \cdots \in E\), then \(\bigcup_{i=1}^{\infty} A_i \in E\)

3. Generated \(\sigma\)-algebra
   (a) \(B\) is a set of sets
   (b) \(\pi(B)\) is the partition generated by \(B\)
   (c) \(\sigma(B) = \sigma(\pi(B)), |\sigma(B)| = 2^{|\pi(B)|}\)

4. Function terminology (validity, domain, co-domain, image, inverse image, range, one-to-one, onto, monotonic)

5. Probability measure \(\wp\)
   Axioms: P1: \(\wp(A) \geq 0, \forall A \in E\)
   P2: \(\wp(\Omega) = 1\) (or \(\wp(\phi) = 0\))
   P3a: \(\wp(A \cup B) = \wp(A) + \wp(B)\), whenever \(A\) and \(B\) are disjoint
   P3b: \(\wp(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \wp(A_i)\), whenever \(A_1, A_2, \cdots\) are disjoint

6. Probability model
   (a) Variable name \(X\): Used to denote the outcome of a particular experiment.
   (b) Sample space \(\Omega\): The set of all possible outcomes.
   (c) Event space \(E\): Set of events, i.e. set of subsets of \(\Omega\). Should be a \(\sigma\)-algebra.
   (d) Probability measure \(\wp : E \to [0, 1]\). Should satisfy all three axioms.

7. Compound experiments
(a) $X = (X_1, X_2, \ldots, X_N)$
(b) $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_N$
(c) $E = \sigma(E_1 \otimes E_2 \otimes \cdots \otimes E_N)$, where $E_1 \otimes E_2 = \{F_1 \times F_2 : F_1 \in E_1, F_2 \in E_2\}$.
(d) No general formula for $\wp$.

8. Conditional probability

(a) Law of total probability: $\wp(A) = \wp(A|B)\wp(B) + \wp(A|B^c)\wp(B^c)$
(b) Bayes’ Rule: $\wp(B|A) = \frac{\wp(A|B)\wp(B)}{\wp(A|B)\wp(B) + \wp(A|B^c)\wp(B^c)}$

9. Independence

10. Combinatorics ($\{\text{ordered sampling, unordered sampling}\} \times \{\text{with replacement, without replacement}\}$)

Problems:

1. Prove P3a from P3b (equivalently, prove S3a from S3b).
2. (Gubner p.51, problem 31) Show that if $B_n$ is a sequence of events for which
   \[ \sum_{n=1}^{\infty} \wp(B_n) < \infty \] (1)
   then
   \[ \wp(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} B_k) = 0 \] (2)
3. When the sample space of a probability model is $\Omega = [0, 1]$, is it possible to take the event space to be the power set of $\Omega$?
4. To prove $N$ events are independent, how many equations should we check?
5. Suppose that $\{E_n, n \geq 1\}$ and $\{F_n, n \geq 1\}$ are increasing sequences of events having limits $E$ and $F$. Show that if $E_n$ is independent of $F_m$ for all $n, m$, then $E$ is independent of $F$. 