Recitation 7

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Topics:

1. Convergence of random variables.

   (a) A sequence \((X_n)\) converges towards \(X\)
   
   i. with probability one, if
   
   \[\varnothing\{w: \lim_{n \to \infty} X_n(w) = X(w)\} = 1\]  \hspace{1cm} (1)
   
   ii. in mean square, if
   
   \[\lim_{n \to \infty} E[|X_n - X|^2] = 0\]  \hspace{1cm} (2)
   
   iii. in probability, if
   
   \[\forall \varepsilon > 0, \lim_{n \to \infty} Pr(|X_n - X| < \varepsilon) = 1\]  \hspace{1cm} (3)
   
   iv. in distribution, if
   
   \[\forall a \in \mathbb{R}, \lim_{n \to \infty} Pr(X_n \leq a) = Pr(X \leq a)\]  \hspace{1cm} (4)

(b) Relations

   \hspace{1cm} Convergence with probability one
   \hspace{2cm} \downarrow
   \hspace{2cm} Convergence in probability
   \hspace{2cm} \uparrow
   \hspace{2cm} Convergence in mean square
   \hspace{1cm} \longrightarrow
   \hspace{1cm} Convergence in distribution

2. Discrete random variables

3. Conditional probability

4. Conditional expectation

Problems:

1. Prove that if a sequence of random variables \((X_n)\) converges to \(X\) with probability one, then it converges to \(X\) in probability.
2. (*02 fall, midterm 2, problem 7) Let $X_1, X_2, \ldots$ be a sequence of random variables defined on the underlying probability experiment $(\Omega, E, \varphi)$, where $\Omega = [0, 1]$, $E =$ Borel $\sigma$-algebra on $\Omega$, and $\varphi$ is defined by $\varphi((a, b)) = b - a$ for $0 \leq a < b \leq 1$. In particular, let

$$X_n(w) = \begin{cases} n, & 0 \leq w \leq \frac{1}{n} \\ 0, & \text{else} \end{cases} \quad (6)$$

(a) Do the $X_n$’s converge to 0 in probability?
(b) Do the $X_n$’s converge to 0 with probability one?
(c) Do the $X_n$’s converge to 0 in mean square?

3. Let $X \sim \text{Binomial}(n, p)$, find $E[X^2]$.

4. (Gubner p.66, problem 66) Apple crates are supposed to contain only red apples, but occasionally a few green apples are found. Assume that the number of red apples and the number of green apples are independent Poisson random variables with parameters $\rho$ and $\gamma$, respectively. Given that a crate contains a total of $k$ apples, find the conditional probability that none of the apples is green.

5. (Gubner p.120, problem 76) Let $X \sim \text{Bernoulli}(2/3)$, and suppose that given $X = i$, $Y \sim \text{Poisson}(3(i+1))$. Find $E[(X + 1)Y^2]$

6. Prove $\text{Var}(X) = E_Y[\text{Var}_X(X|Y)] + \text{Var}_Y(E_X[X|Y])$. 

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