EECS 530 Homework #1
Assigned: Thursday 8th September, 2011
Due: Thursday 15th September, 2011

Some Useful Vector Calculus Identities

$$V = V(x, y, z)$$

$$\mathbf{A} = \mathbf{A}(x, y, z) = A_x(x, y, z) + A_y(x, y, z) + A_z(x, y, z)$$

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{x}(A_y B_z - A_z B_y) + \mathbf{y}(A_z B_x - A_x B_z) + \mathbf{z}(A_x B_y - A_y B_x)$$

$$\nabla = \mathbf{x}\frac{\partial}{\partial x} + \mathbf{y}\frac{\partial}{\partial y} + \mathbf{z}\frac{\partial}{\partial z}$$

$$\text{Gradient: } \nabla V = \mathbf{x}\frac{\partial V}{\partial x} + \mathbf{y}\frac{\partial V}{\partial y} + \mathbf{z}\frac{\partial V}{\partial z}$$

$$\text{Laplacian: } \nabla^2 V = \frac{\partial^2 V}{\partial^2 x} + \frac{\partial^2 V}{\partial^2 y} + \frac{\partial^2 V}{\partial^2 z}$$

$$\text{Divergence: } \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\text{Curl: } \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \mathbf{x}(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}) + \mathbf{y}(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}) + \mathbf{z}(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x})$$

These only apply in Cartesian coordinates. You will later learn how to define them in any general coordinate system. Note that ∇ in the last 2 identities can be thought of as a vector and simply applying dot or cross product.

Question 1

Prove Stokes Theorem, assuming a continuous surface S, bounded by a continuous contour C and a differential vector field A.

$$\oint_C \mathbf{A}\cdot d\mathbf{l} = \iint_S \nabla\times\mathbf{A}\cdot d\mathbf{S}$$

Note: the Divergence theorem can be proved with similar approach. The

Question 2

Prove the following vector identities in general coordinate system:

$a)\ \nabla\times(\nabla f) = 0$
$b)\ \nabla\cdot(\nabla\times\mathbf{A}) = 0$
$c)\ \nabla(fg) = g\nabla f + f\nabla g$

Question 3

Prove the following useful vector identities:

$a)\ \nabla\times(\nabla\times\mathbf{E}) = \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\boldsymbol{E}$ hint: use back-cab rule: $A\times(B\times C) = B(A.C) - C(A.B)$
$b)\ \nabla\times(f\boldsymbol{A}) = f\nabla\times\boldsymbol{A} + (\nabla f)\times\boldsymbol{A}$ hint: use $\nabla\times(f\vec{a}) = \nabla f\times\vec{a}$ ($\vec{a}$ is a const. vector)
$c)\ \nabla\cdot(f\boldsymbol{A}) = f\nabla\cdot\boldsymbol{A} + (\nabla f)\cdot\boldsymbol{A}$ hint: use $\nabla.(f\vec{a}) = \nabla f.\vec{a}$ ($\vec{a}$ is a const. vector
$d)\ \nabla\cdot(\mathbf{E}\times\mathbf{H}) = \boldsymbol{H}\cdot(\nabla\times\boldsymbol{E}) - \boldsymbol{E}\cdot(\nabla\times\boldsymbol{H})$

Question 4

Starting from Maxwell's equations, derive the circuit law for capacitors $I = C\frac{dV}{dt}$, and the circuit law for the inductors $V = L\frac{dI}{dt}$.

Question 5

One of the most important non-elementary functions used in Electromagnetic theory is the Bessel function. This arises in the solution of the Bessel differential equation, a differential equation very common to problems with cylindrical geometries. Appendix D in Harrington gives a quick review on Bessel functions. The Bessel differential equation is:

$$x^2\frac{d^2y}{d^2x}+x\frac{dy}{dx}+(x^2-n^2)y=0$$

A complete solution of this differential equation would require two linearly independent solutions. One solution, known as the Bessel function of the first kind, is of the form $J_n(x)$. When n is not an integer, $J_{-n}(x)$ can be used as the 2nd solution. However, for integer n, $J_{-n}(x)=(-1)^n J_n(x)$. In that case, a 2nd solution, known as the Bessel function of the second kind, $Y_n(x)$, can be used. Y_n can be related to J_n with the following formula.

$$Y_n(x)=\frac{J_n(x)\cos(n\pi)-J_{-n}(-x)}{\sin(n\pi)}$$

Several definitions exist for $J_n(x)$. Here are a few that will be of use to us.

$$J_n(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{(m)!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2m+n}=\frac{i^{-n}}{\pi}\int_0^{\pi}\cos(n\phi)e^{ix\cos\phi}d\phi=\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\phi-n\phi)d\phi=\frac{i^{-n}}{2\pi}\int_0^{2\pi}e^{ix\cos\phi}e^{in\phi}d\phi$$

a) Show that $J_n(x)$ satisfies the Bessel differential equation for **integer n**.

b) Show that $J_{-n}(x)=(-1)^n J_n(x)$.

c) Show that $\frac{d}{dx}J_n(x)=\frac{1}{2}\left[J_{n-1}(x)-J_{n+1}(x)\right]$.

d) Show that $J_n(x)=\frac{x}{2n}\left[J_{n-1}(x)+J_{n+1}(x)\right]$.

e) Show that $\frac{d}{dx}\left[x^n J_n(x)\right]=x^n J_{n-1}(x)$.

Computer Assignment

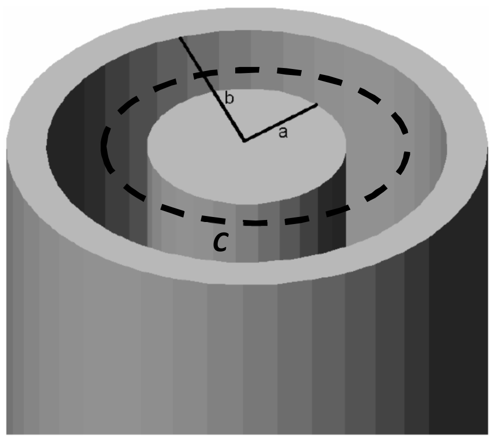
Write a Matlab code to visualize the Bessel functions of the first and second kind as a function of x. The resulting figures (one figure for each kind) should include the cases of n=0, 1, 2, and 3. In a table, show the first four zero crossings of Bessel functions for the same values of n.

Question 6

A coaxial cable, as shown in figure below, is given with inner conductor of radius "*a*" and air dielectric from a < r < b. Two sets of currents are present on the inner conductor.

(1) $J_0 = \hat{\mathbf{z}} e^{-100(1-\frac{r}{a})} \cos(\omega t - kz) \qquad 0 < \mathrm{r} < \mathrm{a} \qquad (\mathrm{A/m^2})$

(2) Volumetric charge density ρ_0 $(\mathrm{C/m^3})$, distributed uniformly over the cross-section of the inner conductor, moving with a constant velocity $u\hat{\mathbf{z}}$. These charges only exist on the inner conductor.



a) Determine the total current flowing on the inner conductor (units should be A), considering both current distributions on the inner conductor.

b) Determine the total linear charge density on the inner conductor (units should be C/m).

c) Using the cylindrical symmetry of the problem, determine the magnetic field H_0 in the region a < r < b.

d) State which, if either, of the two current components will produce a radial electric field.

e) Assume that now your reference contour, "***C***", moves with velocity $\mathbf{v} = v_0\hat{\mathbf{z}}$, where v_0 is comparable in magnitude to u but much less than c. Compute the magnetic field H as seen by this reference contour.

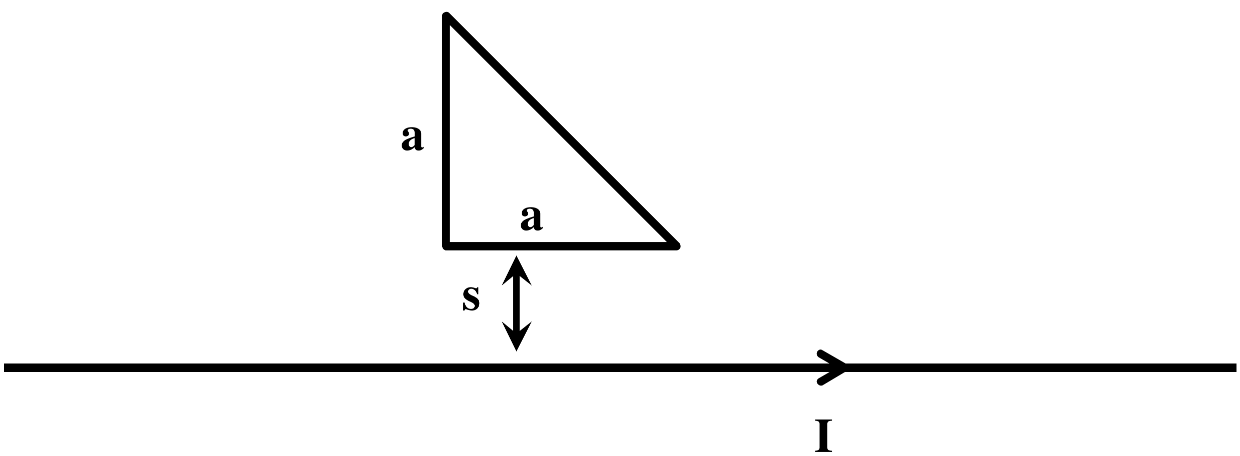
Question 7

For the following electric field in a source free region (ρ=0 and J=0), find the corresponding magnetic field. Does this field satisfy Maxwell's equations?

$$\boldsymbol{E} = (5xy\hat{x} + z\hat{y})\cos(\omega t + \phi_0)$$

Question 8

A triangular loop of wire (side *a*) lies on a table, at distance s from a very long straight wire, which carries a current *I*, as shown below.



a) Find the flux of **B** through the loop.
b) If someone now pulls the loop directly away from the wire, at speed *v*, what electromotive force V_{emf} is generated? In what direction (clockwise or counterclockwise) does the current flow?
c) What if the loop is pulled to the *right* at speed *v* , instead of away?
Now a **slowly varying** current *I(t)* flows through the wire.
d) Repeat question b) for this case.