EECS 530 '06
HW # 1 SOLNS

1. a) $\bar{A}\times(\bar{B}\times\bar{C}) = \bar{A}\times\left[\hat{x}(B_yC_z - B_zC_y) + \hat{y}(B_zC_x - B_xC_z) + \hat{z}(B_xC_y - B_yC_x)\right]$

$$= \left\{\begin{array}{l} \hat{x}\,(A_yB_xC_y - A_yB_yC_x - A_zB_zC_x + A_zB_xC_z) + \\ \hat{y}\,(A_zB_yC_z - A_zB_zC_y - A_xB_xC_y + A_xB_yC_x) + \\ \hat{z}\,(A_xB_zC_x - A_xB_xC_z - A_yB_yC_z + A_yB_zC_y) \end{array}\right\}$$

$$= (\bar{A}\cdot\bar{C})\bar{B} - (\bar{A}\cdot\bar{B})\bar{C}$$

Substitute $\bar{A} = \bar{B} = \nabla$, $\bar{C} = \bar{E}$.

Only ambiguous step is the undefined $(\nabla\cdot\nabla)\bar{E}$ which can be expanded quickly and verified to be the same as $\nabla^2\bar{E}$. ∎

b) $\nabla\cdot(\bar{E}\times\bar{H}) = \nabla\cdot\left\{\begin{array}{l}\hat{x}\,(E_yH_z - E_zH_y) + \\ \hat{y}\,(E_zH_x - E_xH_z) + \\ \hat{z}\,(E_xH_y - E_yH_z)\end{array}\right\}$

$$= \left\{\begin{array}{l} \left(\frac{\partial E_y}{\partial x}H_z + E_y\frac{\partial H_z}{\partial x} - \frac{\partial E_z}{\partial z}H_y - E_z\frac{\partial H_y}{\partial x}\right) + \\ \left(\frac{\partial E_z}{\partial y}H_x + E_z\frac{\partial H_z}{\partial y} - \frac{\partial E_x}{\partial y}H_z - E_x\frac{\partial H_z}{\partial y}\right) + \\ \left(\frac{\partial E_x}{\partial z}H_y + E_z\frac{\partial H_y}{\partial z} - \frac{\partial E_y}{\partial z}H_x - E_y\frac{\partial H_x}{\partial z}\right) \end{array}\right\}$$

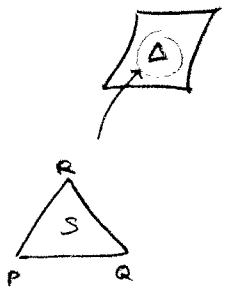
$$= \bar{H}\cdot\left\{\hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{y}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\right\}$$

$$- \bar{E}\cdot\left\{\hat{x}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \hat{y}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + \hat{z}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\right\}$$

$$= \bar{H}\cdot(\nabla\times\bar{E}) - \bar{E}\cdot(\nabla\times\bar{H})$$

∎

2. Any continuous surface can be approximated by a series of infinitesimally small triangles.
All sides of the triangle are coplanar
Let $\hat{t}_1, \hat{t}_2$ be any two sides
we define $\hat{n} = \dfrac{\hat{t}_1 \times \hat{t}_2}{|\hat{t}_1 \times \hat{t}_2|}$



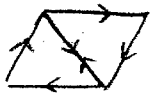
Then, $\oint_C \bar{A} \cdot d\bar{l}$

$= \int_{q_1}^{q_2} \left(A_{t_1} \frac{dt_1}{dq} + A_{t_2} \frac{dt_2}{dq} + A_n \cancel{\frac{dn}{dq}}^{\to 0} \right) dq$. q is a variable used to integrate over the contours of the triangle

$= \oint_C \left(A_{t_1} dt_1 + A_{t_2} dt_2 \right)$

$= \iint_S \frac{d}{dt_2} A_{t_1} - \frac{d}{dt_1} A_{t_2}$ (2D Green's Thm)

$= \iint_S (\nabla \times A) \cdot \hat{n}\, dA = \iint_S (\nabla \times A) \cdot d\bar{S}$ since $d\bar{S} = \hat{n}\, dA$
continuous

Since any surface can be meshed with triangles, and for any two triangles,  , the common side will be cancelled in the line integral due to being integrated in two opposite directions, this method works for any continuous surface. ■

3. Assume closed volume V bounded with surfaces S_1, S_2.

$$\iiint_V \frac{\partial A_x}{\partial x}\, dV = \iint_{y,z} \left(\int_{S_1}^{S_2} \frac{\partial A_x}{\partial x}\, dx \right) dy\, dz$$

$$= \iint_{y,z} A_x(S_2(y,z), y, z) - A_x(S_1(y,z), y, z)\, dy\, dz$$

$$= \iint_{S_2} A\, \hat{x} \cdot \hat{n}\, dA + \iint_{S_1} A_x\, \hat{x} \cdot \hat{n}\, dA$$

$$= \oiint_S A_x \cdot d\bar{S}$$

By extension, $\iiint_V \nabla \cdot \bar{A}\, dV = \oiint_S \bar{A} \cdot d\bar{S}$ ■

4. a) $\nabla \cdot \nabla \times \bar{A} = \lim_{\Delta V \to 0} \frac{1}{V} \oiint_S \nabla \times \bar{A} \cdot d\bar{S}$

$$\oiint_S \nabla \times \bar{A} \cdot d\bar{S} = \iint_{S_1} \nabla \times \bar{A} \cdot d\bar{S} + \iint_{S_2} \nabla \times \bar{A} \cdot dS_2$$

choose arbitrary contour C_1 such that splits closed surface into S_1, S_2.

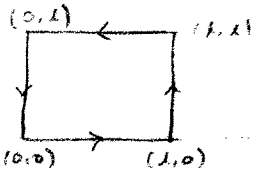
$$\Rightarrow \iint_{S_1} \nabla \times \bar{A} \cdot d\bar{S} + \iint_{S_2} \nabla \times \bar{A} \cdot d\bar{S}$$

(Stokes) $= \oint_{C_1} \bar{A} \cdot d\bar{l} + \oint_{C_2} \bar{A} \cdot d\bar{l}$ but C_2 is the same contour as C_1, in the opposite direction.

$$= \oint_{C_1} \bar{A} \cdot d\bar{l} - \oint_{C_1} \bar{A} \cdot d\bar{l}$$

$$= 0 \quad \Rightarrow \quad \nabla \cdot \nabla \times \bar{A} = 0 \quad \blacksquare$$

b) Consider square S



$$\oint_C \nabla \varphi \cdot d\bar{l}$$

$$= \int_0^L (\nabla \varphi|_{y=L} - \nabla \varphi|_{y=0}) \cdot d\bar{x} + \int_0^L (\nabla \varphi|_{x=0} - \nabla \varphi|_{x=L}) \cdot d\bar{y}$$

$$= \varphi|_{y=L}|_{x=0}^{L} - \varphi|_{y=0}|_{x=0}^{L} + \varphi|_{x=0}|_{y=0}^{L} - \varphi|_{x=L}|_{y=0}^{L} \quad \text{(FTC)}$$

$$= 0$$

Since $(\nabla \times \nabla \varphi) \cdot \hat{n} = \lim_{\Delta S \to 0} \frac{\oint \nabla \varphi \cdot d\bar{l}}{\Delta S}$

and showed $\oint \nabla \varphi \cdot d\bar{l} = 0$ for arbitrary contour $\Rightarrow \nabla \times \nabla \varphi = 0$ $\blacksquare$

5. a) Use $J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m+n}}{m!\,(n+m)!}$ since $\Gamma(n+m+1) = (n+m)!$

$$x^2 \frac{\partial^2 y}{\partial x^2} + x \frac{\partial y}{\partial x} + (x^2 - n^2)\, y = 0$$

$$\rightarrow \sum_{m=0}^{\infty} \frac{(-1)^m (2m+n)(2m+n-1)}{m!\,(m+n)!} \left(\frac{x}{2}\right)^{2m+n}$$

$$+ \sum_{m=0}^{\infty} \frac{(-1)^m (2m+n)}{m!\,(m+n)!} \left(\frac{x}{2}\right)^{2m+n}$$

$$+ \sum_{m=0}^{\infty} \frac{(-1)^m (-n^2)}{m!\,(m+n)!} \left(\frac{x}{2}\right)^{2m+n}$$

$$+ \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+n)!} \left(\frac{x}{2}\right)^{2m+n} x^2$$

$$\left(\frac{x}{2}\right)^{2m+n} \cdot x^2 = \left(\frac{x}{2}\right)^{2m+n} \left(\frac{x}{2}\right)^2 \cdot 4 = 4 \left(\frac{x}{2}\right)^{2(m+1)+n}$$

Last term $\sum_{m=0}^{\infty} \frac{(-1)^m (4) \left(\frac{x}{2}\right)^{2(m+1)+n}}{m!\,(n+m)!}$

$$= \sum_{m=0}^{\infty} \frac{(-1)^{m+1} (-4) \left(\frac{x}{2}\right)^{2(m+1)+n}}{(m+1)!\,(n+m+1)!} (m+1)(n+m+1)$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^m (-4) \left(\frac{x}{2}\right)^{2m+n}}{m!\,(n+m)!} (m)(n+m)$$

$\Rightarrow$ Sum coefficient from all terms

$m=0$: $(n)(n-1) + (n) - n^2 = 0$

$m > 0$: $(2m+n)(2m+n-1) + (2m+n) - n^2 - 4(m)(n+m)$

$= 4m^2 + 4mn - 2m + n^2 - n + 2m + n - n^2 - 4mn - 4m^2 = 0$

Thus, $J_n(x)$ satisfies the Bessel differential equation. ■

b) $J_{-n}(x) = \frac{i^n}{\pi} \int_0^{\pi} \cos(-n\phi)\, e^{ix\cos\phi}\, d\phi = \frac{i^n}{\pi} \int_0^{\pi} \cos(n\phi)\, e^{ix\cos\phi}\, d\phi$

but $J_n(x) = \frac{i^{-n}}{\pi} \int_0^{\pi} \cos(n\phi)\, e^{ix\cos\phi}\, d\phi$

$$\frac{J_{-n}(x)}{J(x)} = \frac{i^n}{i^{-n}} = i^{2n} = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases} = (-1)^n$$

$$\rightarrow \quad J_{-n}(x) = (-1)^n J_n(x)$$

■

c) $$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\phi - n\phi)\, d\phi$$

$$J_n'(x) = \frac{1}{\pi}\int_0^{\pi} \sin(x\sin\phi - n\phi)\cdot \sin\phi \, d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{1}{2}\left(\cos(x\sin\phi - (n-1)\phi) - \cos(x\sin\phi - (n+1)\phi)\right) d\phi$$

$$= \frac{1}{2}\left(J_{n-1}(x) + J_{n+1}(x)\right)$$ ■

d) $$\frac{\partial}{\partial n}\left(J_{n-1}(x) + J_{n+1}(x)\right) = \frac{\partial}{\partial n}\left[\frac{i^{-(n-1)}}{\pi}\int_0^{2\pi} e^{ix\cos\phi} e^{i(n-1)\phi}\, d\phi + \frac{i^{-(n+1)}}{\pi}\int_0^{2\pi} e^{ix\cos\phi} e^{i(n+1)\phi} d\phi\right]$$

$$= \frac{\partial}{\partial n}\frac{i^{-n}}{\pi}\int_0^{2\pi} i\, e^{ix\cos\phi}\left(e^{i(n-1)\phi} - e^{i(n+1)\phi}\right) d\phi$$

$$= \frac{\partial}{\partial n}\frac{i^{-n}}{\pi}\int_0^{2\pi} e^{ix\cos\phi} e^{in\phi}\sin\phi \, d\phi$$

$$dv = e^{ix\cos\phi}(-\sin\phi)\, d\phi \rightarrow v = \frac{1}{ix} e^{ix\cos\phi}$$

$$u = -e^{in\phi} \rightarrow du = -in\, e^{in\phi}$$

$$\rightarrow \frac{\partial}{\partial n}\frac{i^{-n}}{\pi}\left(\left(-e^{in\phi}\right)\left(\frac{1}{ix} e^{ix\cos\phi}\right)\Big|_0^{2\pi} - \int_0^{2\pi} -\frac{n}{x} e^{ix\cos\phi} e^{in\phi}\, d\phi\right)$$

$$= \frac{i^{-n}}{\pi}\int_0^{2\pi} e^{ix\cos\phi} e^{in\phi}\, d\phi = J_n(x)$$ ■

e) $$\frac{d}{dx}\left(x^n J_n(x)\right) = nx^{n-1} J_n(x) + x^n \frac{1}{2}\left(J_{n-1}(x) - J_{n+1}(x)\right)$$

$$= \frac{x^n}{2}\left(\frac{2n}{x} J_n(x) + J_{n-1}(x) - J_{n+1}(x)\right)$$

$$= \frac{x^n}{2}\left(J_{n-1}(x) + J_{n+1}(x) + J_{n-1}(x) - J_{n+1}(x)\right)$$

$$= x^n J_{n-1}(x)$$ ■

6. a) $\bar{E} = \frac{\cos(\omega t - k|\bar{r}-\bar{r}'|)}{4\pi|\bar{r}-\bar{r}'|}\hat{z}$, $\hat{z} = \hat{R}\cos\theta - \hat{\theta}\sin\theta$

let $R = |\bar{r}-\bar{r}'|$

$$\nabla\cdot\bar{D} = \nabla\cdot\epsilon\bar{E} = \epsilon\,\nabla\cdot\left(\frac{\cos(\omega t - kR)}{4\pi R}\left(\hat{R}\cos\theta - \hat{\theta}\sin\theta\right)\right)$$

$$= \epsilon\left(\frac{\cos(\omega t-kR) - R\sin(\omega t-kR)}{4\pi R^2}\right)\cos\theta -$$

$$2\epsilon\,\frac{\cos(\omega t-kR)}{4\pi R}\cos\theta \neq 0$$

$\Rightarrow$ Does not satisfy source-free Maxwell's equations ∎

b) $\bar{E} = J_0(kr)\cos(\omega t)\,\hat{z}$

$$-\frac{\partial\bar{B}}{\partial t} = \nabla\times E = \hat{\phi}\left(-k\cdot\frac{1}{2}\left(J_{-1}(kr) - J_1(kr)\right)\cos(\omega t)\right)$$

$$= \hat{\phi}\left(k\,J_1(kr)\cos(\omega t)\right)$$

$$\bar{B} = -\hat{\phi}\,\frac{k}{\omega}J_1(kr)\sin(\omega t) \rightarrow \bar{H} = -\hat{\phi}\,\frac{k}{\omega\mu}J_1(kr)\sin(\omega t)$$

$$\nabla\cdot\bar{B} = \frac{1}{r}\frac{\partial}{\partial\phi}(B_\phi) = 0$$

$$\nabla\cdot\bar{D} = \epsilon\left(\nabla\cdot\bar{E}\right) = \epsilon\left(\frac{\partial E_z}{\partial z}\right) = 0$$

$$\nabla\times\bar{H} = -\hat{z}\cdot\frac{1}{r}\left(\frac{\partial}{\partial r}\left(\frac{k}{\omega\mu}r\,J_1(kr)\sin(\omega t)\right)\right)$$

$$= -\hat{z}\cdot\frac{1}{r}\left(\frac{k}{\omega\mu}\sin(\omega t)\right)\left(r\,J_0(kr)\right)$$

$$= -\hat{z}\cdot\frac{k^2}{\omega\mu}\sin(\omega t)\,J_0(kr) = \frac{\partial\bar{D}}{\partial t}$$

$$\rightarrow \bar{D} = \hat{z}\,\frac{k^2}{\omega^2\mu}\cos(\omega t)\,J_0(kr) \qquad \frac{k^2}{\omega^2\mu} = \frac{\omega^2\mu\epsilon}{\omega^2\mu} = \epsilon$$

$$\rightarrow \bar{D} = \hat{z}\,\epsilon\cos(\omega t)\,J_0(kr)$$

$$\rightarrow \bar{E} = \hat{z}\cos(\omega t)\,J_0(kr)$$

$\Rightarrow$ Satisfies Maxwell's Equations ∎

7. a) $J_0 = \hat{z}\, e^{-100(1-\frac{r}{a})} \cos(\omega t - kz)$

$$I_0 = \iint_S J_0 \cdot d\bar{s}$$

$$= \int_0^{2\pi} d\phi \int_0^a r\,dr \left(I e^{-100(1-\frac{r}{a})} \cos(\omega t - kz) \right) \hat{z}$$

$$= 2\pi \cos(\omega t - kz) \cdot \int_0^a r e^{-100(1-\frac{r}{a})}\, dr\ \hat{z}$$

$$u = r \rightarrow du = dr$$

$$dv = e^{-100(1-\frac{r}{a})}\, dr \rightarrow v = \frac{a}{100} e^{-100(1-\frac{r}{a})}$$

$$= 2\pi \cos(\omega t - kz) \cdot \left(\frac{a^2}{100} - \int \frac{a}{100} e^{-100(1-\frac{r}{a})} dr \right) \hat{z}$$

$$= 2\pi \cos(\omega t - kz) \cdot \left(\frac{a^2}{100} - \frac{a^2}{10000} \left(1 - e^{-100^0}\right) \right) \hat{z}$$

$$= 2\pi \cos(\omega t - kz) \cdot a^2 \cdot \frac{99}{10000}\, \hat{z}$$

Volumetric charge density ρ_0 distribute uniformly over cylinder cross section => linear charge density $\rho_l = \pi a^2 \cdot \rho_0$ (c/m) and $I = \rho_0 \pi a^2 \cdot v\, \hat{z}$

$$\rightarrow I_{total} = \left(2\pi \cos(\omega t - kz) \cdot \frac{99a^2}{10000} + \rho_0 \pi a^2 v \right) \hat{z}$$

b) continuity equation: $\frac{\partial \rho_v}{\partial t} = -\nabla \cdot \bar{J}$ =>

$$\frac{\partial \rho_l}{\partial t} = -\nabla \cdot I_{total}$$

$$= -2\pi k \sin(\omega t - kz) \frac{99a^2}{10000}$$

$$\rho_l = \frac{2\pi k}{\omega} \cos(\omega t - kz) \frac{99a^2}{10000} + \text{constant}$$

$$\text{constant} = \pi a^2 \rho_0$$

$$\rightarrow \rho_l = \frac{2\pi k}{\omega} \cos(\omega t - kz) \frac{99a^2}{10000} + \pi a^2 \rho_0$$

c) See Ulaby 5-2.1 for derivation

$$\bar{H}_0 = \hat{\phi} \frac{I}{2\pi r} = \frac{\hat{\phi}}{2\pi r} \left(2\pi \cos(\omega t - kz) \cdot \frac{99a^2}{10000} + \rho_0 \pi a^2 v \right)$$

d) Both components of I will produce $\bar{E}$, and just like $\bar{H}$, one will be time varying while the other is static.

More specifically, $\bar{E} = \frac{\bar{D}}{\epsilon} = \frac{1}{\epsilon}\frac{\rho_l}{2\pi a}\hat{r}$

$$= \epsilon^{-1}\frac{1}{2\pi a}\left(\frac{2\pi u}{\omega}\cdot\cos(\omega t - kz)\frac{99a^2}{10000} + \pi a^2\rho_0\right)\hat{r}$$

e) Due to motion of reference, a displacement current is generated that needs to be taken into account.

$$\oint_c \bar{H}\cdot d\bar{l} = I_{total} - \oint_c \bar{v}\times\bar{D}\cdot d\bar{l}, \qquad \bar{D} = \frac{\rho_l}{2\pi a}\hat{r} \qquad d\bar{l} = \hat{\Phi}$$

$$\bar{v} = v_0\hat{z}$$

$$\bar{H} = \bar{H_0} - \bar{v}\times\bar{D}$$

$$= \bar{H_0} - \frac{v_0}{2\pi a}\left(\frac{2\pi u}{\omega}\cos(\omega t - kz)\frac{99a^2}{10000} + \pi a^2\rho_0\right)\hat{\Phi}$$

$$= \frac{\hat{\Phi}}{2\pi a}\left(2\pi\cos(\omega t - kz)\cdot\frac{99a^2}{10000}\left(1-\frac{v}{c}\right) + \pi a^2\rho_0(u - v_0)\right)$$

↑ Lorentzian factor from special relativity

↑ Due to relative motion between charges and reference contour.