1) The definition of $\nabla \times \vec{A}$ is $(\nabla \times \vec{A}) \cdot \Delta \vec{S} = \oint_C \vec{A} \cdot d\vec{\ell}$ when $\Delta S \to 0$ $\quad (\Delta \vec{S} = \Delta S \cdot \vec{n})$

For an arbitrary surface S, we can subdivide it into many small differential areas.

Therefore

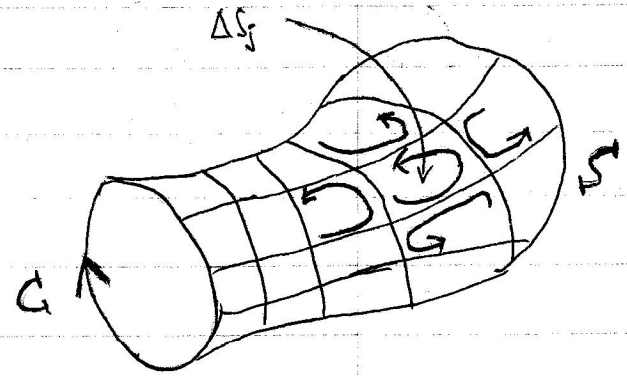
$$\lim_{\Delta S_j \to 0} \sum_{j=1}^{N} (\nabla \times \vec{A})_j \cdot \Delta \vec{S}_j = \int_S (\nabla \times \vec{A}) \cdot d\vec{S}$$

Now we can sum up the line integrals.

Since the common part of the contours of two adjacent elements is traversed in opposite directions by two contours, the net contribution of all common parts is zero and only the contribution from the external contour C bounding the area S remains.

$$\lim_{\Delta S_j \to 0} \sum_{j=1}^{N} (\nabla \times \vec{A})_j \cdot \Delta \vec{S}_j = \lim_{\Delta S_j \to 0} \sum_{j=1}^{N} \oint_C \vec{A}_j \cdot d\vec{\ell} =$$

$$= \oint_C \vec{A} \cdot d\vec{\ell}$$



2) The definition of divergence is $(\nabla \cdot \vec{A})\,\Delta V = \oint_S \vec{A} \cdot d\vec{S}$ when $\Delta V \to 0$

(S is the closed surface over the small differential volume ΔV)

For an arbitrary volume V, we can subdivide it into many small differential volumes. Therefore

$$\lim_{\Delta V_j \to 0} \sum_{j=1}^{N} (\nabla \cdot \vec{A})|_j \, \Delta V_j = \iiint_V \nabla \cdot \vec{A} \, dV$$

With the definition of divergence we get

$$\iiint_V \nabla \cdot \vec{A} \, dV = \lim_{\Delta V_j \to 0} \left(\sum_{j=1}^{N} \oint_{S_j} \vec{A}_j \cdot d\vec{S} \right)$$

On the right hand side, the contributions from the internal surfaces of adjacent elements will cancel each other. Hence the contribution of the right hand side is only due to the external surface.

$$\lim_{\Delta V_j \to 0} \left(\sum_{j=1}^{N} \oint_{S_j} \vec{A}_j \cdot d\vec{S} \right) = \oint_S \vec{A} \cdot d\vec{S}$$

3) a) Use definition of $\nabla \times \vec{A}$:

$$\nabla \times (\nabla f) \cdot \hat{n} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_C \nabla f \, d\vec{\ell}$$

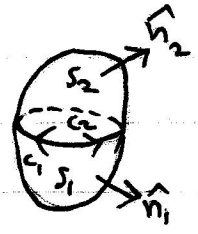
with gradient theorem $\int_L \nabla f \, d\vec{\ell} = f(q) - f(p)$

where q and p are the endpoints of the curve

$$\Rightarrow \oint_C \nabla f \, d\vec{\ell} = 0 \quad \text{for closed curve } C$$

$$\Rightarrow \nabla \times (\nabla f) = 0 \quad \square$$

b) Use definition of $\nabla \cdot \vec{A}$:



$$\nabla \cdot (\nabla \times \vec{A}) = \lim_{\Delta V \to 0} \oiint_S (\nabla \times \vec{A}) \, d\vec{S}$$

Choose arbitrary contour that splits the closed surface S into S_1 and S_2

$$\oiint_S (\nabla \times \vec{A}) \, d\vec{S} = \iint_{S_1} (\nabla \times \vec{A}) \, d\vec{S} + \iint_{S_2} (\nabla \times \vec{A}) \, d\vec{S} = \quad \text{(Stokes theorem)}$$

$$= \oint_{C_1} \vec{A} \cdot d\vec{\ell} + \oint_{C_2} \vec{A} \cdot d\vec{\ell} = \oint_{C_1} \vec{A} \cdot d\vec{\ell} - \oint_{C_1} \vec{A} \cdot d\vec{\ell} = 0$$

$$\Rightarrow \nabla \cdot (\nabla \times \vec{A}) = 0 \quad \square$$

3) c) $$\nabla\times(f\vec{A}) = \begin{pmatrix} \partial_y f A_z - \partial_z f A_y \\ \partial_z f A_x - \partial_x f A_z \\ \partial_x f A_y - \partial_y f A_x \end{pmatrix} =$$

$$= \begin{pmatrix} (\partial_y f) A_z + f\,\partial_y A_z - (\partial_z f) A_y - f\,\partial_z A_y \\ (\partial_z f) A_x + f\,\partial_z A_x - (\partial_x f) A_z - f\,\partial_x A_z \\ (\partial_x f) A_y + f\,\partial_x A_y - (\partial_y f) A_x - f\,\partial_y A_x \end{pmatrix} =$$ ← kgroup

$$= f\,\nabla\times\vec{A} + (\nabla f)\times\vec{A}$$

d) $$\nabla\cdot(f\vec{A}) = \partial_x(fA_x) + \partial_y(fA_y) + \partial_z(fA_z) =$$

$$= (\partial_x f) A_x + f\,\partial_x A_x + (\partial_y f) A_y + f\,\partial_y A_y + (\partial_z f) A_z + f\,\partial_z A_z =$$

kgroup ↘
$$= (\nabla f)\cdot\vec{A} + f\,\nabla\cdot\vec{A}$$

f) $$\nabla\cdot(\vec{E}\times\vec{H}) = \nabla\cdot\begin{pmatrix} E_y H_z - E_z H_y \\ E_z H_x - E_x H_z \\ E_x H_y - E_y H_x \end{pmatrix} =$$

$$= (\partial_x E_y) H_z + E_y(\partial_x H_z) - (\partial_x E_z) H_y - E_z(\partial_x H_y) +$$
$$+ (\partial_y E_z) H_x + E_z(\partial_y H_x) - (\partial_y E_x) H_z - E_x(\partial_y H_z) +$$
$$+ (\partial_z E_x) H_y + E_x(\partial_z H_y) - (\partial_z E_y) H_x - E_y(\partial_z H_x) =$$

kgroup ↘↘
$$= \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}\circ\begin{pmatrix} \partial_y E_z - \partial_z E_y \\ \partial_z E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{pmatrix} - \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}\circ\begin{pmatrix} \partial_y H_z - \partial_z H_y \\ \partial_z H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{pmatrix} =$$

$$= \vec{H}\circ(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{H})$$

3) a) use short notation $\frac{\partial}{\partial x} E \equiv E^x$

$$\nabla(\nabla\cdot\vec{E}) = \nabla\left(E_x^{\,x} + E_y^{\,y} + E_z^{\,z}\right) = \begin{pmatrix} E_x^{\,xx} + E_y^{\,yx} + E_z^{\,zx} \\ E_x^{\,xy} + E_y^{\,yy} + E_z^{\,zy} \\ E_x^{\,xz} + E_y^{\,yz} + E_z^{\,zz} \end{pmatrix}$$

$$\nabla^2\vec{E} = \begin{pmatrix} E_x^{\,xx} + E_x^{\,yy} + E_x^{\,zz} \\ E_y^{\,xx} + E_y^{\,yy} + E_y^{\,zz} \\ E_z^{\,xx} + E_z^{\,yy} + E_z^{\,zz} \end{pmatrix}$$

$$\Rightarrow \nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E} = \begin{pmatrix} E_y^{\,yx} + E_z^{\,zx} - E_x^{\,yy} - E_x^{\,zz} \\ E_x^{\,xy} + E_z^{\,zy} - E_y^{\,xx} - E_y^{\,zz} \\ E_x^{\,xz} + E_y^{\,yz} - E_z^{\,xx} - E_z^{\,yy} \end{pmatrix} \qquad (1)$$

$$\nabla\times(\nabla\times\vec{E}) = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} E_z^{\,y} - E_y^{\,z} \\ E_x^{\,z} - E_z^{\,x} \\ E_y^{\,x} - E_x^{\,y} \end{pmatrix} =$$

$$= \begin{pmatrix} E_y^{\,xy} - E_x^{\,yy} - E_x^{\,zz} + E_z^{\,xz} \\ E_z^{\,yz} - E_y^{\,zz} - E_y^{\,xx} + E_x^{\,yx} \\ E_x^{\,zx} - E_z^{\,xx} - E_z^{\,yy} + E_y^{\,zy} \end{pmatrix} \qquad (2)$$

If derivations can be interchanged, i.e. $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} E_y\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} E_y\right)$

$\Rightarrow (1) = (2)$

4) a) $\vec{r} = \begin{pmatrix} x-x' \\ y-y' \\ z-z' \end{pmatrix}$

$$\nabla(r^2) = \nabla\left((x-x')^2 + (y-y')^2 + (z-z')^2\right) =$$
$$= \hat{x}\,2(x-x') + \hat{y}\,2(y-y') + \hat{z}\,2(z-z') =$$
$$= 2\vec{r}$$

b) $\nabla\left(\frac{1}{r}\right) = \nabla\left(\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-1/2}\right) =$

$$= \begin{pmatrix} 2(x-x') \\ 2(y-y') \\ 2(z-z') \end{pmatrix} \cdot \left(-\tfrac{1}{2}\right) \cdot \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-3/2} =$$

$$= -\frac{\vec{r}}{r^3}$$

c) $\nabla(r^n) = \nabla\left(\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{n/2}\right)$

$$= \frac{n}{2} \cdot \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{\frac{n}{2}-1} \cdot 2 \cdot \begin{pmatrix} x-x' \\ y-y' \\ z-z' \end{pmatrix} =$$
$$= n \cdot r^{n-2} \cdot \vec{r}$$

5) a) use $J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{x}{2}\right)^{2m+n}}{m!\,(n+m)!}$ $\quad \Gamma(n+m+1) = (n+m)!$ for integer n, m

and substitute in diff. equation

$$\sum_{m=0}^{\infty} \frac{(-1)^m (2m+n)(2m+n-1)}{m!\,(n+m)!} \left(\frac{x}{2}\right)^{2m+n} + \sum_{m=0}^{\infty} \frac{(-1)^m (2m+n)}{m!\,(n+m)!} \left(\frac{x}{2}\right)^{2m+n} +$$

$$+ \sum_{m=0}^{\infty} \frac{(-1)^m (-n^2)}{m!\,(n+m)!} \left(\frac{x}{2}\right)^{2m+n} + \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(n+m)!} \left(\frac{x}{2}\right)^{2m+n} \cdot x^2 = 0$$

with $\left(\frac{x}{2}\right)^{2m+n} \cdot x^2 = \left(\frac{x}{2}\right)^{2m+n+2} \cdot 4$

we get for the last term

$$\sum_{m=0}^{\infty} \frac{(-1)^m 4}{m!\,(n+m)!} \left(\frac{x}{2}\right)^{2(m+1)+n} = \sum_{m=0}^{\infty} \frac{(-1)^{m+1} (-4)}{(m+1)!\,(n+m+1)!} (m+1)(n+m+1) \left(\frac{x}{2}\right)^{2(m+1)}$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^m (-4)}{m!\,(n+m)!} \left(\frac{x}{2}\right)^{2m+n} \cdot m \cdot (n+m)$$

now sum coefficients for all terms:

$m = 0$: $\quad \frac{n(n-1)}{n!} + \frac{n}{n!} - \frac{n^2}{n!} = 0$ ✓

$m \geq 1$: $\quad \frac{(-1)^m}{m!\,(n+m)!} \left[(2m+n)(2m+n-1) + (2m+n) - n^2 - 4m(n+m) \right] =$

$$= \frac{(-1)^m}{m!\,(n+m)!} \left[4m^2 + 4mn - 2m + n^2 - n + 2m + n - n^2 - 4mn - 4m^2 \right] =$$

$$= 0 \quad ✓$$

5) b) $J_{-n}(x) = \frac{i^n}{\pi}\int_0^{\pi}\cos(-n\phi)\,e^{ix\cos\phi}\,d\phi = \frac{i^n}{\pi}\int_0^{\pi}\cos(n\phi)\,e^{ix\cos\phi}\,d\phi$

but $J_n(x) = \frac{i^{-n}}{\pi}\int_0^{\pi}\cos(+n\phi)\,e^{ix\cos\phi}\,d\phi$

$\Rightarrow \frac{J_{-n}(x)}{J_n(x)} = \frac{i^n}{i^{-n}} = i^{2n} = (i^2)^n = (-1)^n$

c) $J_n(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\varphi - n\varphi)\,d\varphi$

$\Rightarrow J_n'(x) = \frac{1}{\pi}\int_0^{\pi}\sin(x\sin\varphi - n\varphi)\sin\varphi\,d\varphi =$ ← sine addition formulas

$= \frac{1}{\pi}\int_0^{\pi}\frac{1}{2}\left[\cos(x\sin\varphi - (n-1)\varphi) - \cos(x\sin\varphi - (n-1)\varphi)\right]d\varphi =$

$= \frac{1}{2}\left(J_{n-1}(x) + J_{n+1}(x)\right)$

d) $\frac{x}{2n}\left(J_{n-1}(x) + J_{n+1}(x)\right) = \frac{x}{2n}\left[\frac{i^{(-n+1)}}{\pi}\int_0^{2\pi} e^{ix\cos\varphi}\,e^{i(n-1)\varphi}\,d\varphi + \right.$

$\left. + \frac{i^{(-n-1)}}{\pi}\int_0^{2\pi} e^{ix\cos\varphi}\,e^{i(n+1)\varphi}\,d\varphi\right] =$

$= \frac{x}{2n}\,\frac{i^{-n}}{\pi}\int_0^{2\pi} i\,e^{ix\cos\varphi}\left(e^{i(n-1)\varphi} - e^{i(n+1)\varphi}\right)d\varphi =$

$= \frac{x}{2n}\cdot\frac{i^{-n}}{\pi}\int_0^{2\pi} 2\,e^{ix\cos\varphi}\,e^{in\varphi}\sin\varphi\,d\varphi = \circledast$

$dV = -in\,e^{in\varphi}\,d\varphi$

now $U = -e^{in\varphi}$ and $dV = e^{ix\cos\varphi}(+\sin\varphi)\,d\varphi \Rightarrow V = \frac{1}{ix}\,e^{ix\cos\varphi}$

$\Rightarrow \circledast = \frac{x}{n}\cdot\frac{i^{-n}}{\pi}\left[UV\Big|_0^{2\pi} - \int_0^{2\pi} V\,dU\right] = \frac{x}{n}\cdot\frac{i^{-n}}{\pi}\left[\left(-e^{in\varphi}\right)\frac{1}{ix}\,e^{ix\cos\varphi}\Big|_0^{2\pi} - \right.$ ($-e^{in\varphi}$ → 0)

$\left. - \int_0^{2\pi} -\frac{n}{x}\,e^{ix\cos\varphi}\,e^{in\varphi}\,d\varphi\right] = \frac{i^{-n}}{\pi}\int_0^{2\pi} e^{ix\cos\varphi}\,e^{in\varphi}\,d\varphi = J_n(x)$

5) a) $\frac{d}{dx}\left(x^n J_n(x)\right) = n x^{n-1} J_n(x) + x^n \cdot \frac{1}{2}\left(J_{n-1}(x) - J_{n+1}(x)\right) =$

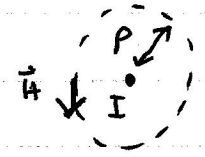
$= \frac{x^n}{2}\left(\frac{2n}{x} J_n(x) + J_{n-1}(x) - J_{n+1}(x)\right) =$

$= \frac{x^n}{2}\left(J_{n-1}(x) + J_{n+1}(x) + J_{n-1}(x) - J_{n+1}(x)\right) =$

$= x^n J_{n-1}(x)$

6) a) flux $\Phi = \int_A \vec{B} \cdot d\vec{A}$

med $\vec{B} = \mu \vec{H}$: $\oint_C \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{S} = I$



$\vec{H} = H \hat{\varphi}$ $\quad d\vec{l} = \rho \, d\varphi \, \hat{\varphi}$

$\Rightarrow \oint_C \vec{H} d\vec{l} = \int_0^{2\pi} H \hat{\varphi} \cdot \rho \, d\varphi \, \hat{\varphi} = H\rho \int_0^{2\pi} d\varphi = H\rho 2\pi$

$\Rightarrow \vec{H} = \frac{I}{2\pi\rho} \hat{\varphi} \Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi\rho} \hat{\varphi}$

$\Rightarrow \Phi = \int_A \frac{\mu_0 I}{2\pi\rho} \hat{\varphi} \, d\vec{A} = \int_s^{s+a} \int_0^a \frac{\mu_0 I}{2\pi\rho} \, d\rho \, dx =$

$= \int_s^{s+a} \frac{\mu_0 I a}{2\pi\rho} \, d\rho = \frac{\mu_0 I a}{2\pi} \left[\ln \rho \right]_s^{s+a} = \frac{\mu_0 I}{2\pi} a \left[\ln(s+a) - \ln s \right]$

b) $V_{ind} = \frac{d\Phi}{dt}$ $\quad \Phi(t) = \frac{\mu_0 I a}{2\pi} \left[\ln(s+a+vt) - \ln(s+vt) \right]$

$\Rightarrow \frac{d\Phi}{dt} = \frac{\mu_0 I a}{2\pi} \cdot v \cdot \left[\frac{1}{s+a+vt} - \frac{1}{s+vt} \right] = V_{ind}$

current flows clockwise

c) in this case $\Phi = \text{const} \Rightarrow V_{ind} = 0$

d) again $V_{ind} = \frac{d\Phi}{dt}$ but now

$\Phi(t) = \frac{\mu_0 a I(t)}{2\pi} \left[\ln(s+a+vt) - \ln(s+vt) \right]$

$\Rightarrow \frac{d\Phi}{dt} = \frac{\mu_0 a}{2\pi} \left\{ I(t) \cdot v \cdot \left[\frac{1}{s+a+vt} - \frac{1}{s+vt} \right] + \frac{dI(t)}{dt} \left[\ln(s+a+vt) - \ln(s+vt) \right] \right\}$

$+ \frac{dI(t)}{dt} \left[\ln(s+a+vt) - \ln(s+vt) \right] \Big\}$

7) I) $\nabla \times \vec{B} = 0$

satisfied since $\vec{B} = 0$ (✓)

II) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = 0$ $\qquad \vec{E} = \begin{pmatrix} E_\varphi \\ E_\theta \\ E_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{4\pi\varepsilon} \frac{q}{r^2} \Theta(vt-r) \end{pmatrix} = E_r(r)\,\hat{r}$

$\Rightarrow \nabla \times \vec{E} = 0$ since $E_r \neq f(\varphi) \neq f(\theta)$

(✓)

III) $\nabla \cdot \vec{D} = \rho \Rightarrow \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$

$$\nabla \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 E_r\right) = \frac{1}{r^2} \frac{\partial}{\partial r}\left(-\frac{1}{4\pi\varepsilon} q\,\Theta(vt-r)\right) =$$

$$= \frac{q}{4\pi\varepsilon_0} \frac{1}{r^2} \delta(vt-r)$$

$$\Rightarrow \rho = \frac{q}{4\pi r^2} \delta(vt-r) \qquad (\checkmark)$$

derivative of step function $\Theta(x)$ is dirac function $\delta(x)$

now at given point $\vec{r}_1$, no field at beginning, appears at $t = \frac{|\vec{r}_1|}{v}$

$\Rightarrow$ apply Gauß' law in integral form inside charge sphere

$$\oint_S \varepsilon_0 \vec{E} \cdot d\vec{S} = \iiint_V \rho\, dV = \int_0^{\pi} \int_0^{2\pi} \frac{-q}{4\pi r^2} r^2 \sin\theta\, d\varphi\, d\theta =$$

$$= -q$$

This is valid for every $0 < r < \frac{v}{t}$

$\Rightarrow$ additional charge at center with $-q$

$$\Rightarrow \rho = \frac{q}{4\pi r^2} \delta(vt-r) - q\,\delta(r)$$

IV) $\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \Rightarrow \vec{j} = -\frac{\partial \vec{D}}{\partial t} = -\frac{q}{4\pi r^2} \hat{r} \frac{\partial}{\partial t} \Theta(vt-r) =$

$$= \frac{qv}{4\pi r^2} \delta(vt-r) \cdot \hat{r} \qquad (\checkmark)$$

explanation: expanding ((inwardly) spherical balloon containing charge q ... shows $-q$ at center