

1 The Field Equations

Electromagnetism, as the structure of the word implies, deals with certain laws of physics with interrelated electric and magnetic fields. Electromagnetic phenomena are observed only when the two field quantities are time-varying. The time-varying electric and magnetic fields interactions phenomena were discovered by Michael Faraday (1791–1867) and James Clerk Maxwell (1831–1879). Faraday's significant discovery was based on experimentation and Maxwell's was based on mathematical deduction. Faraday's extensive experimental work was motivated by the belief that every cause and effect has its converse. That is, if electricity can produce a magnetic field, the phenomenon discovered by Oersted, then a magnetic field should be able to produce an electric field.

The fundamental laws of electricity and magnetism are encapsulated by Maxwell's equations:[1]

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \text{Modified Ampère's law} \tag{1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{Faraday's law} \tag{2}$$

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{v}} \qquad \text{Gauss' law of electricity} \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{Gauss' law of magnetism} \tag{4}$$

where

$\mathbf{H}(\mathbf{r}, t)$	magnetic field intensity	A/m
$\mathbf{E}(\mathbf{r}, t)$	electric field intensity	V/m
$\mathbf{D}(\mathbf{r}, t)$	electric flux density	$\mathrm{C/m^2}$
$\mathbf{B}(\mathbf{r}, t)$	magnetic flux density	$\mathrm{W/m^2}$
$\rho_{\mathrm{v}}(\mathbf{r}, t)$	volumetric charge density	$\mathrm{C/m^3}$

$\mathbf{r}$ is the position vector for an ordinary point in the medium. Here "ordinary point" refers to a point where in its neighborhood the physical properties of the medium are continuous. In other words, the small medium around $\mathbf{r}$ is considered to be "homogeneous." If the variations of the physical properties of a medium is abrupt the vector field quantities may also vary sharply. That is, the transition of field vectors across a surface where change in material properties is abrupt may be discontinuous. The nature of these discontinuities will be investigated in detail later.

Equations (1) and (2) in addition to the *Law of Conservation of Charges* are the necessary and sufficient set of equations for determining the field quantities.

The phenomena of electricity and magnetism are explained by the presence of static and moving charges. Basically charges are considered the source of electromagnetic fields without which the field quantities $(\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B})$ cannot exist.

[1] James Clerk Maxwell, *A Treatise on Electricity and Magnetism*, Constable and Co., London, 1873.

1.1 Charge Density

The quantized nature of charges is well established. A charge quantum is equal to the absolute value of the charge of an electron given by

$$\text{electron charge } e = -1.6 \times 10^{-19} \text{ C}$$

However, Maxwell's equations describe large-scale phenomena, i.e., the macroscopic element of volume must contain a large number of atoms and molecules. In this case charge of a macroscopic particle can assume any real number. Suppose the amount of charge contained in volume element ΔV is Δq, then the volumetric charge density is defined as

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta q}{\Delta V} \tag{5}$$

In the strict sense (5) does not define a continuous function of position since ΔV can't approach zero without limit.

1.2 Current Density

From a macroscopic point of view, any ordered motion of charge constitutes a current. Hence the current can be represented by a vector quantity whose direction is the direction of motion and its magnitude is proportional to the velocity of the charge. To define this vector quantity more precisely, current density (distribution), associated with a point in space is defined. Current density (distribution) is characterized by a vector field **J** as depicted in Fig. 1.

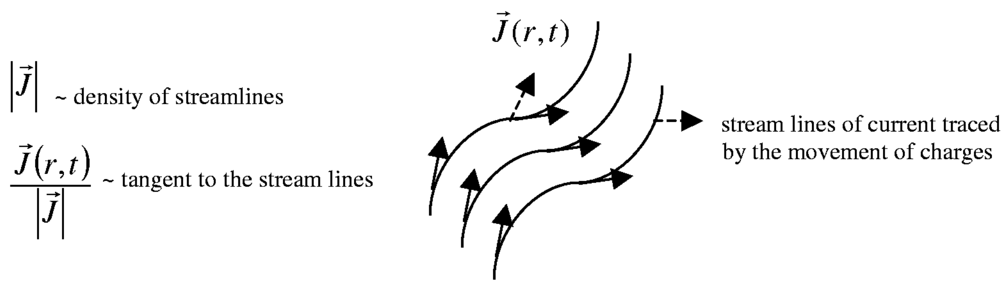


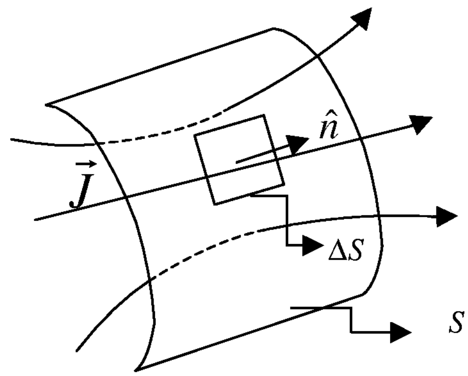
Fig. 1: Stream lines of electric current in a medium, where the intensity of the current density is represented by the density of the stream lines the unit vector tangent to the lines indicate the direction of flow.

To better quantify current density let us onsider a differential surface ΔS whose unit normal is denoted by $\hat{n}$ as shown in Fig. 2. Referring to Fig. 2, if ΔI represents the total current crossing the differential area ΔS, then **J** is defined so that

$$\Delta I = \mathbf{J} \cdot \Delta \mathbf{S}$$

It is now obvious that the total current crossing an arbitrary surface S can be computed from:

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} \tag{6}$$



$$\Delta \mathbf{S} = \Delta S\,\hat{n}$$

Fig. 2: A surface intersecting stream lines of an electric current in a medium and a differential surface at a point on the surface for characterizing the current density.

1.3 Point Form of Law of Conservation of Charge

The law of conservation of charges states that the net value of charge in a closed system remains constant. This indicates that if there are a certain number of positive and negative charges in an enclosed surface, nothing can be done to create an excess amount of any kind of charges or annihilate only one type of these charges. To make any changes, charges have to be taken away from or brought into the system.

Now suppose the surface S is a closed surface. Define $\hat{n}$ as a unit normal to the surface drawn *outward*. According to the law of conservation of charges,

$$I = \oiint_S \mathbf{J} \cdot d\mathbf{s} = -\frac{d}{dt}\,Q = -\frac{d}{dt}\int_V \rho\, dv,$$

where Q is the total charge enclosed. If the surface is stationary,

$$\oiint \mathbf{J} \cdot d\mathbf{s} = \iiint\limits_V -\frac{\partial \rho}{\partial t}\, dv \tag{7}$$

According to the divergence theorem,

$$\oiint \mathbf{J} \cdot d\mathbf{s} = \iiint\limits_V \nabla \cdot \mathbf{J}\, dv \tag{8}$$

In limit where v is very small $\iiint_{\Delta V} \nabla \cdot \mathbf{J}\, dv = \nabla \cdot \mathbf{J}\, \Delta V$. This gives a definition for the divergence operation:

$$\nabla \cdot \mathbf{J} \triangleq \lim_{\Delta v \to 0} \frac{1}{\Delta v} \oiint \mathbf{J} \cdot d\mathbf{s}$$

Using (8) in (7),

$$\iiint\limits_V \nabla \cdot \mathbf{J}\, dv = \iiint\limits_V -\frac{\partial \rho}{\partial t}\, dv \tag{9}$$

Since (9) has to be valid for all arbitrary volumes,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{10}$$

"Law of conservation of charge in the neighborhood of a point," also known as the *equation of continuity*.

1.4 Interdependence of Maxwell's Equations

Equations (3) and (4) of Maxwell's equations (Gauss' laws) *are not independent*.

Noting that

$$\nabla \cdot \nabla \times \mathbf{A} = 0, \qquad \forall \mathbf{A}$$

from (1)

$$\nabla \cdot \mathbf{J} + \frac{\partial}{\partial t} \nabla \cdot \mathbf{D} = 0 \tag{11}$$

Substituting (10) in (11)

$$\frac{\partial}{\partial t} (\nabla \cdot \mathbf{D} - \rho) = 0,$$

which implies $(\nabla \cdot \mathbf{D} - \rho)$ must be a constant. If the field and charge over their history have vanished over the entire space then the constant must be zero.

$$\nabla \cdot \mathbf{D} = \rho,$$

which is Gauss' law of electricity.

Also from (2) taking divergence from both sides

$$\frac{\partial}{\partial t} \nabla \cdot \mathbf{B} = 0,$$

from which again it can be concluded that

$$\nabla \cdot \mathbf{B} = 0$$

In summary, Faraday's law, modified Ampère's law, and the equation of continuity are the sufficient set of equations for characterization of field quantities.

1.5 Integral Form of Maxwell's Equations

Curl of a vector field quantity whose components and their first derivation are continuous is defined by

$$(\nabla \times \mathbf{A}) \cdot \hat{n} = \lim_{\Delta s \to 0} \frac{\oint \mathbf{A} \cdot d\bar{\ell}}{\Delta s} \tag{12}$$

All three components of $\nabla \times \mathbf{A}$ can be obtained once $\hat{n}$ is aligned with the coordinate unit vectors.

* Directions $\hat{n}$ and differential length $d\bar{\ell}$ follow the right-hand rule ($d\bar{\ell}$ along fingers and $\hat{n}$ along the right-hand thumb).

Stokes' theorem is a natural extension of the curl definition. For a regular closed contour C and any arbitrary surface S bounded by C over which the components of $\mathbf{A}$ and their first derivative are continuous, Stokes' theorem states

$$\iint_S \nabla \times \mathbf{A} \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\bar{\ell} \tag{13}$$

Taking surface integral from both sides of (1) and applying (13) it can easily be shown that

$$\oint_C \mathbf{H} \cdot d\bar{\ell} = \iint_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s}$$

$\partial\mathbf{D}/\partial t$ has the unit of A/m^2 and is also known as the *displacement current*. If S is stationary, then

$$\oint_C \mathbf{H} \cdot d\bar{\ell} = I + \frac{d}{dt} \iint\limits_S \mathbf{D} \cdot d\mathbf{s}, \tag{14}$$

where I is the total conduction current going through a surface defined by C.

Equation (14) is more general than (1), which allows for the surface S and contour C themselves to be time-varying. In fact (14) is the physical formulation from which (1) is derived.

In a similar manner the integral form of Faraday's law can be obtained and is given by

$$\oint_C \mathbf{E} \cdot d\bar{\ell} = -\frac{d}{dt} \iint\limits_S \mathbf{B} \cdot d\mathbf{s}, \tag{15}$$

which states that the induced voltage around a closed contour is equal to the negative time rate of change of linking flux.

Taking volume integral from both sides of (3),

$$\iiint\limits_V \nabla \cdot D \, dv = \iiint\limits_V \rho \, dv = Q$$

apply the divergence theorem:

$$\oiint \mathbf{D} \cdot d\mathbf{s} = Q \qquad \text{Gauss' law}$$

Similarly it can be shown that

$$\oiint \mathbf{B} \cdot d\mathbf{s} = 0$$

which can be interpreted as "flux lines of magnetic flux density are continuous."