

## 2 Integral form of Maxwell's equations and time-varying surfaces

As discussed before the general forms of Maxwell's modified Ampère's and Faraday's laws are given by

$$\oint_C \mathbf{H} \cdot d\bar{\ell} = I + \frac{d}{dt} \iint_S \mathbf{D} \cdot d\mathbf{s} \quad (1)$$

$$\oint_C \mathbf{E} \cdot d\bar{\ell} = -\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{s} \quad (2)$$

In situations where both the field quantities and  $s$  are varying with time explicit expressions for

$$\tilde{V} = \frac{d}{dt} \iint_{S(t)} \mathbf{B} \cdot d\mathbf{s} \quad (3)$$

$$\tilde{I} = \frac{d}{dt} \iint_{S(t)} \mathbf{D} \cdot d\mathbf{s} \quad (4)$$

are needed. To evaluate (3) or (4) we start with the definition of time derivative. For example for (3)

$$\tilde{V} = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left\{ \iint_{S(t+\Delta t)} \mathbf{B}(t+\Delta t) \cdot d\mathbf{s} - \iint_{S(t)} \mathbf{B}(t) \cdot d\mathbf{s} \right\}$$

Noting that

$$\begin{aligned} \mathbf{B}(t+\Delta t) &\simeq \mathbf{B}(t) + \frac{\partial \mathbf{B}(t)}{\partial t} \Delta t \\ \tilde{V}_1 &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \iint_{S(t+\Delta t)} \left( \frac{\partial \mathbf{B}}{\partial t} \Delta t \right) \cdot d\mathbf{s} = \iint_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} \end{aligned} \quad (5)$$

The second component to be evaluated is

$$\tilde{V}_2 = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left\{ \iint_{S(t+\Delta t)} \mathbf{B}(t) \cdot d\mathbf{s} - \iint_{S(t)} \mathbf{B}(t) \cdot d\mathbf{s} \right\} \quad (6)$$

Consider a moving surface geometry shown in Fig. 2.1, which shows progress of  $S(t)$  in the time interval  $\Delta t$ .

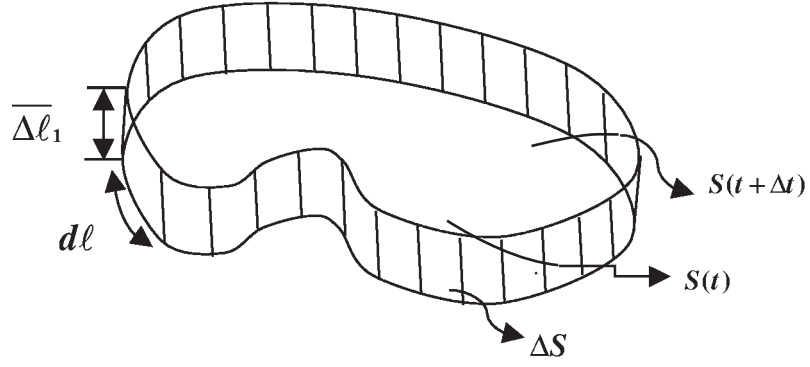


Figure 2.1. A moving surface in space. Shown is the surface change in a small time interval  $\Delta t$ .

Every point on the contour moves by

$$d\bar{l}_1 = \mathbf{v} \Delta t$$

where  $\mathbf{v}$  is the local speed of the point. As the contour moves it traces a surface on the side. We denote this surface by  $\Delta S$ . Let us also denote the closed surface formed by  $S(t)$ ,  $S(t + \Delta t)$  and  $\Delta S$  by  $S_0$ . Therefore the second component given by (6) can be written as

$$\tilde{V}_2 = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left\{ \oint_{S_0} \mathbf{B}(t) \cdot d\mathbf{s} - \iint_{\Delta S} \mathbf{B}(t) \cdot d\mathbf{s} \right\} \quad (7)$$

where the differential surface  $d\mathbf{s}$  for the side surface  $\Delta S$  is given by

$$d\mathbf{s} = d\bar{l} \times \Delta\bar{l}_1 = d\bar{l} \times \mathbf{v} \Delta t$$

Thus

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \iint_{\Delta S} \mathbf{B}(t) \cdot d\mathbf{s} &= \oint_C \mathbf{B}(t) \cdot d\bar{l} \times \mathbf{v}(t) \\ &= \oint_C [\mathbf{v}(t) \times \mathbf{B}(t)] \cdot d\bar{l} \end{aligned} \quad (8)$$

Also

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \oint_{S_0} \mathbf{B} \cdot d\mathbf{s} &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \iiint \nabla \cdot \mathbf{B} \, dv \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \iint_{S(t)} (\nabla \cdot \mathbf{B}) \mathbf{v} \Delta t \cdot d\mathbf{s} \\ &= \iint_{S(t)} (\nabla \cdot \mathbf{B}) \mathbf{v} \cdot d\mathbf{s} \end{aligned} \quad (9)$$

Since  $\nabla \cdot \mathbf{B} = 0$ , the integral term given by (9) vanishes, and (2) now can be written as

$$\oint_C \mathbf{E} \cdot d\bar{l} = \iint_S -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} + \oint_C (\mathbf{v} \times \mathbf{B}) \cdot d\bar{l} \quad (10)$$

The first term is referred to as the “transformer induction” and the second term is known as the “motional induction.”

It is very important to note that in order to arrive at a non-vanishing motional induction term, *the contour of the surface  $S$  has to be moving*. That is, if the contour is fixed and  $S(t)$  is still a function time there will be no motional induction. Physically the contour of the integral (circuit) has to “cut” the flux lines of the magnetic flux density in order to generate “electromotive potential.”

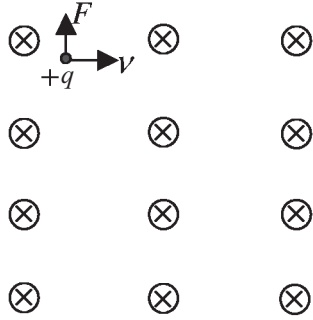
This phenomenon can physically be explained using the Lorentz force on a charged particle  $q$

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \quad (11)$$

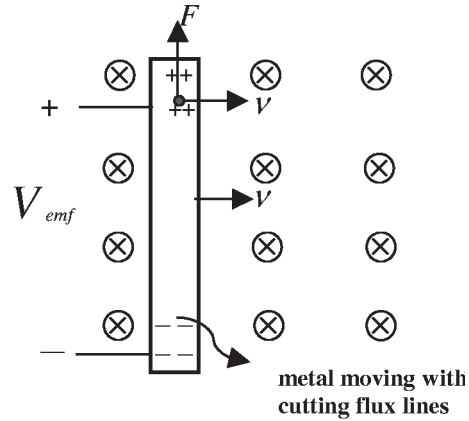
Since electric field is defined in terms of force per unit charge, the equivalent induced electric field due to moving charges can be defined

$$\begin{aligned} \mathbf{E}_{\text{emf}} &= \frac{\mathbf{F}_{\text{emf}}}{q} = \mathbf{v} \times \mathbf{B} \\ V_{\text{emf}} &= \oint \mathbf{E}_{\text{emf}} \cdot d\bar{\ell} = \oint (\mathbf{v} \times \mathbf{B}) \cdot d\bar{\ell} \end{aligned} \quad (12)$$

Equation (12) is consistent with (10).



(a) a positive moving charge in a magnetic field



(b) a metallic rod moving in a magnetic field which produces electromotive voltage across the bar

In a similar manner (1) can be written as

$$\oint \mathbf{H} \cdot d\bar{\ell} = I + \iint_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} + \iint_S (\nabla \cdot \mathbf{D}) \mathbf{v} \cdot d\mathbf{s} - \oint_C (\mathbf{v} \times \mathbf{D}) \cdot d\bar{\ell}$$

Noting that  $\nabla \cdot \mathbf{D} = \rho$

$$\oint_C \mathbf{H} \cdot d\bar{\ell} = I + \iint_S \rho \mathbf{v} \cdot d\mathbf{s} + \iint_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} - \oint_C (\mathbf{v} \times \mathbf{D}) \cdot d\bar{\ell} \quad (13)$$

in which  $I$  represents the conduction current,  $\rho \mathbf{v}$  represents the drift current,  $\partial \mathbf{D} / \partial t$  represents the displacement current, and  $(\mathbf{v} \times \mathbf{D})$  represents the “motional current.”