ECS 530 **Lecture 3**

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3 Constitutive Relations: Macroscopic Properties of Matter

As shown previously, out of the four Maxwell's equations only the Faraday's and modified Ampère's laws are independent. Considering the fact that the charges and currents are the sources of electromagnetic fields $\mathbf{E}, \mathbf{D}$ and $\mathbf{H}, \mathbf{B}$, in addition to the two independent Maxwell's equations we need to impose two constraints if the system of equations is to be made determinate.

In the most general case, we may consider the relationship among the four field quantities to be of the form of

$$\mathbf{D} = D(\mathbf{E}, \mathbf{H}) \tag{1}$$

$$\mathbf{B} = B(\mathbf{E}, \mathbf{H}) \tag{2}$$

where $D(\cdot)$ and $B(\cdot)$ are some general vector functions dependent upon the material in which the field vector quantities are established.

In the treatment of Maxwell's equations considered henceforth, we shall confine our attention only to small-signal condition where the constitutive relations given by (1) and (2) are *linear*. That is, if $\mathbf{D}_1$ and $\mathbf{B}_1$ result from $\mathbf{E}_1$ and $\mathbf{H}_1$ and $\mathbf{D}_2$ and $\mathbf{B}_2$ are established due to $\mathbf{E}_2$ and $\mathbf{H}_2$, then $\mathbf{D}_1 + \mathbf{D}_2$ and $\mathbf{B}_1 + \mathbf{B}_2$ will be established due to $\mathbf{E}_1 + \mathbf{E}_2$ and $\mathbf{H}_1 + \mathbf{H}_2$.

The constants of proportionality for a material can themselves be a function of position. These materials are referred to as *inhomogeneous* materials, in contrast to materials for which the constants of proportionality are invariant with space, which are called *homogeneous*.

- In the absence of any material (vacuum) the constitutive relations are very simple and are given by

$$\mathbf{D} = \epsilon_0 \mathbf{E}, \qquad \mathbf{B} = \mu_0 \mathbf{H} \tag{3}$$

  where

$\epsilon_0 = 8.85 \times 10^{-12}$	farad/m	free-space permittivity
$\mu_0 = 4\pi \times 10^{-7}$	henry/m	free-space permeability

- Isotropic homogeneous material

  If the physical properties of the medium are the same in all directions (as seen by $\mathbf{E}$ and $\mathbf{H}$) the medium is isotropic. At every point $\mathbf{D}$ is parallel to $\mathbf{E}$ and $\mathbf{B}$ is parallel to $\mathbf{H}$.

$$\mathbf{D} = \epsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H}, \tag{4}$$

  where $\epsilon = \epsilon_r \epsilon_0$ and $\mu = \mu_r \mu_0$

  ϵ_r = relative permittivity (dimensionless)
  μ_r = relative permeability (dimensionless)

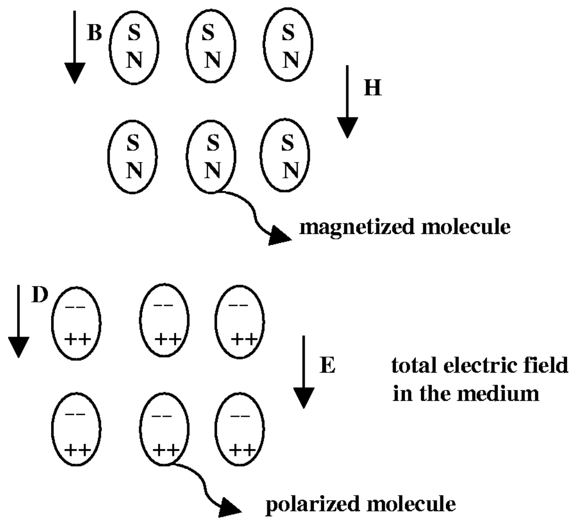

Figure 1: Magnetized and polarized molecules in the presence of external applied electric and magnetic fields. The total field in such a medium is weaker than the external applied field in the absence of the material.

Usually ϵ_{r} and μ_{r} are quantities larger than unity.

In the presence of an external electric and magnetic field the material gets polarized and magnetized respectively. The electric flux density in a medium can be written as

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

where $\mathbf{P}$ is known as *polarization vector* and indicates dipole moment per unit volume, and is related to the electric field by $\mathbf{P} = \epsilon_0 \chi_{\mathrm{e}} \mathbf{E}$ where χ_{e} is the electric susceptibility of the medium. According to (4) the permittivity of the medium is thus given by

$$\epsilon = \epsilon_0 (1 + \chi_{\mathrm{e}}).$$

In a similar manner the magnetic flux density in a magnetic medium can be written as

$$\mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M}$$

where $\mathbf{M}$ is known as *magnetization vector* and provides a measure of induced magnetic dipole moment per unit volume.

Magnetic materials can in general be categorized as:

1. Paramagnetic ($\mu_{\mathrm{r}} > 1$)

2. Diamagnetic ($\mu_{\mathrm{r}} < 1$): induced magnetic moments are parallel to the applied magnetic field.

3. Ferromagnetic ($\mu_{\mathrm{r}} \gg 1$): Indicates spontaneous magnetization in subdomains. Highly non-linear. Characterized by hysteresis (a time-varying phenomenon related to material memory). Above Curie temperature the material becomes paramagnetic.

The phenomenon of ferroelectricity has also been observed for certain materials; for example, barium titanate ($BaTiO_3$) shows ferroelectricity behavior that is marked by very large ϵ_{r}, hysteresis, and non-linearity.

Anisotropic Media

For certain materials properties of the matter vary in different manners along different directions. For these materials vectors $\mathbf{B}$ and $\mathbf{D}$ are not necessarily parallel to $\mathbf{H}$ and $\mathbf{E}$ respectively. In this case the linear relationship between $\mathbf{D}$ and $\mathbf{E}$, and $\mathbf{B}$ and $\mathbf{H}$, are expressed by permittivity and permeability tensors:

$$\mathbf{D} = \overline{\overline{\epsilon}} \cdot \mathbf{E} \qquad \overline{\overline{\epsilon}} = \epsilon_0 \overline{\overline{\epsilon}}_\mathrm{r} = \epsilon_0 \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \tag{5}$$

$$\mathbf{B} = \overline{\overline{\mu}} \cdot \mathbf{H} \qquad \overline{\overline{\mu}} = \mu_0 \overline{\overline{\mu}}_\mathrm{r} \tag{6}$$

A material is called *anisotropic* if either or both permittivity and permeability are tensor quantities. It should be noted that the tensor entries are functions of the coordinate system. For materials with axes of symmetry, the permittivity or permeability tensors become symmetric matrices. In this case the matrix (matrices) are diagonizable with real eigenvalues and eigen vectors that are orthogonal. Rotating the Cartesian coordinate system and making it parallel to the eigen vectors' directions the tensor becomes diagonal:

$$\overline{\overline{\epsilon}} = \epsilon_0 \begin{bmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix} \tag{7}$$

This material is known as *biaxial material*. If there is invariance in any coordinate plane two of the entries of (7) become equal (say $\epsilon_x = \epsilon_y \neq \epsilon_z$). This material is known as a *uniaxial medium*.

Bianisotropic Media

Bianisotropic matter exhibits magnetization under an externally applied electric field and polarization under an externally applied magnetic field in addition to the magnetization and polarization usually obtained from the external fields. The constitutive relationships for these materials are given by

$$\mathbf{D} = \overline{\overline{\epsilon}} \cdot \mathbf{E} + \overline{\overline{\zeta}} \cdot \mathbf{H} \tag{8}$$

$$\mathbf{B} = \overline{\overline{\xi}} \cdot \mathbf{E} + \overline{\overline{\mu}} \cdot \mathbf{H} \tag{9}$$

Dispersive Materials

Except for free space for which the permittivity and permeability are constant functions of time and frequency, all other media, strictly speaking, are frequency dependent. This is due to the fact that all charges that interact with the field quantities have finite mass and thus there is a time delay between formation of polarization and magnetization vectors and the applied external fields.

Under small signal approximation (linear) the polarization vector (dipole moment per unit volume) is linearly proportional to the applied electric field. For an isotropic medium,

$$\mathbf{P} = \epsilon_0 \cdot \chi_\mathrm{e} \mathbf{E} \tag{10}$$

where χ_e is called the *electric susceptibility*. If χ_e is assumed a real constant and independent of frequency, the relationship given by (10) will ignore the delay between $\mathbf{E}$

and $\mathbf{P}$. The correct way of thinking about a linear relationship between $\mathbf{P}$ and $\mathbf{E}$ is a time-domain convolution. Suppose $\chi_{\mathrm{e}}(t)$ represents the impulse response; then

$$\mathbf{P}(t) = \epsilon_0 \int_{-\infty}^{t} \chi_{\mathrm{e}}(t-\tau)\,\mathbf{E}(\tau)\,d\tau$$

The reason for truncation of the integral at t is due to the imposition of the causality condition, i.e., $\chi_{\mathrm{e}}(t) = 0$ for $t < 0$. In this case

$$\begin{aligned}\mathbf{D}(t) &= \epsilon_0 \mathbf{E}(t) + \epsilon_0 \int_{-\infty}^{t} \chi_{\mathrm{e}}(t-\tau)\,\mathbf{E}(\tau)\,d\tau \\ &= \epsilon_0 \mathbf{E}(t) + \epsilon_0 \int_{0}^{\infty} \chi_{\mathrm{e}}(\tau)\,\mathbf{E}(t-\tau)\,d\tau \end{aligned} \tag{11}$$

Taking the Fourier transform from both sides of (11)

$$\begin{aligned}\int_{-\infty}^{+\infty} \mathbf{D}(t)\,e^{i\omega t}\,dt &= \epsilon_0 \int_{-\infty}^{+\infty} \left[\mathbf{E}(t) + \int_{0}^{\infty} \chi_{\mathrm{e}}(\tau)\,\mathbf{E}(t-\tau)\,d\tau\right] e^{i\omega t}\,dt \\ \mathbf{D}(\omega) &= \epsilon_0\,\mathbf{E}(\omega) + \epsilon_0 \int_{0}^{\infty} \chi_{\mathrm{e}}(\tau)\,e^{i\omega\tau} \underbrace{\int_{-\infty}^{+\infty} \mathbf{E}(t-\tau)\,e^{i\omega(t-\tau)}\,dt}_{\mathbf{E}(\omega)}\,d\tau\end{aligned}$$

so

$$\mathbf{D}(\omega) = \epsilon_0 \left(1 + \int_{0}^{\infty} \chi_{\mathrm{e}}(\tau)\,e^{i\omega\tau}\,d\tau\right)\,\mathbf{E}(\omega)$$

Therefore the complex permittivity is given by

$$\begin{aligned}\epsilon(\omega) &= \epsilon_0 \left(1 + \int_{0}^{\infty} \chi_{\mathrm{e}}(\tau)\,e^{i\omega\tau}\,d\tau\right) \\ &= \epsilon_0(1 + \chi_{\mathrm{e}}(\omega))\end{aligned} \tag{12}$$

It is obvious that, in general, for every material the permittivity is a complex number and a function of frequency. Considering a narrow pulse propagating in such a medium, it can be shown that the pulse will spread in time and space as it travels in the medium.

Similar behavior can be shown to be true for magnetic materials by substituting the electric susceptibility with magnetic susceptibility defined by

$$\mathbf{M} = \chi_{\mathrm{m}}(\omega)\,\mathbf{H}$$

The variations of permittivity and permeability with frequency vary significantly for different materials. For most non-polar materials, ϵ and μ can be approximated by constant quantities at microwave and millimeter-wave parts of the spectrum. Near molecular resonances the variations of the constitutive parameters with frequency must be taken into account carefully.

Conducting Media

Another element of the constitutive relations is the relation between the current density and electric field in a medium. In the absence of magnetic fields, a conducting medium is characterized by

$$\mathbf{J}_{\mathrm{c}} = \sigma \mathbf{E} \tag{13}$$

where $\mathbf{J}_c$ is the conduction current (to be differentiated from impressed source current), $\mathbf{E}$ is the electric field in the medium, and σ is the medium conductivity. Equation (13) is known as the *point form of Ohm's law*. In the presence of a magnetic field the direction of conduction current is no longer parallel to $\mathbf{E}$. This phenomenon is known as the *Hall effect*.

Within a conducting medium ($\sigma > 0$) there can be no accumulation of charges. If an initial charge density $\rho_0(\mathbf{r})$ is established in a conducting medium, according to the equation of continuity we have

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t}$$

by Ohm's law

$$\nabla \cdot \sigma E = -\frac{\partial \rho}{\partial t} \tag{14}$$

and by Gauss' law

$$\sigma \nabla \cdot E = \frac{\sigma}{\epsilon} \nabla \cdot D = \frac{\sigma}{\epsilon} \rho \tag{15}$$

Using (14) in (15), we have

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon} \rho = 0 \qquad \Longrightarrow \qquad \rho = \rho_0 e^{-(\sigma/\epsilon)t}$$

That is, at any point in the medium charges vanish exponentially. That is, the positive and negative charges recombine, or move away from each other to be accumulated at the surface of bounded conducting medium.

The Lorentz Dielectric Model

A classical model for complex dielectric constant of materials was developed H. A. Lorentz at the turn of the twentieth century. This model is based on a simple oscillatory mechanical system in which bound electrons are allowed to move around stationary ions under the driving force of applied electromagnetic fields. In this model each molecule is considered to be independent of other modules within the matter. That is, the motion of electrons of a molecule does not influence the others and vice versa. Also under small signal approximation the electrostatic force on displaced electron clouds around an ion is described by a linear relation. Electron collision is described by a damping coefficient in an equivalent mechanical system shown in Fig. 2 The equation of motion for the electron

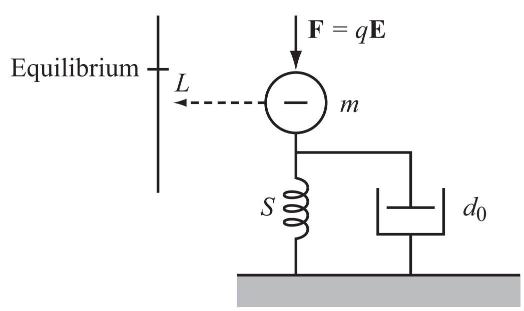

Figure 2: An equivalent mechanical model describing motion of bound electrons around ions subject to an external electric field.

clouds of mass m under local electric force field E is given by

$$m\,\frac{d^2\mathbf{L}}{dt^2} + d_0\,\frac{d\mathbf{L}}{dt} + S\mathbf{L} = q\,\mathbf{E}(t)$$

where S is the spring constant representing linearized electrostatic force and d_0 is the damping coefficient. Assuming time harmonic excitation, i.e., $\mathbf{E}(t) = \text{Re}[\widetilde{E}e^{-i\omega t}]$. The phasor form of displacement is given by

$$\widetilde{L} = \frac{(q/m)\widetilde{E}}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

where $\omega_0^2 = S/m$ is the natural (resonant) frequency of the system and $\gamma = d_0/m$ is the damping factor. Now assuming there are N independent polarized molecules per unit volume, then the polarization is given by

$$\mathbf{P} = Nq\mathbf{L} = \frac{\omega_\text{p}^2}{\omega_0^2 - \omega^2 - i\gamma\omega}\,\epsilon_0\mathbf{E}$$

where $\omega_\text{p}^2 = Nq^2/m\epsilon_0$ is defined as the plasma frequency. Recalling that $\mathbf{P} = \epsilon_0\chi_\text{e}\mathbf{E}$ and $\epsilon = \epsilon_0(1+\chi_\text{e})$, the dielectric constant of the medium is given by

$$\epsilon = \epsilon_0\left[1 + \frac{\omega_\text{p}^2}{\omega_0^2 - \omega^2 - i\gamma\omega}\right] \tag{16}$$

whose real and imaginary parts can be written as

$$\epsilon' = \epsilon_0\left[1 + \frac{\omega_\text{p}^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}\right] \tag{17}$$

$$\epsilon'' = \epsilon_0\left[\frac{\omega_\text{p}^2\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}\right] \tag{18}$$

It is interesting to note that near resonance $\omega \sim \omega_0$ the real part of ϵ can become less than unity or even a negative number. The maximum value of ϵ'' occurs around $\omega = \omega_0$ and its peak value is approximately equal to $\omega_\text{p}^2/8\omega_0$. Figure 3 shows the plot of ϵ' and ϵ'' as a function of frequency ω for some typical values of ω_0, ω_p and γ.

At frequencies much less than the resonant frequency ($\omega \ll \omega_0$) it can easily be shown that

$$\epsilon' = \epsilon_0\left(1 + \frac{\omega_\text{p}^2}{\omega_0^2}\right) \tag{19}$$

$$\epsilon'' = \epsilon_0\,\frac{\gamma\omega_\text{p}^2\omega}{\omega_0^4} \tag{20}$$

This indicates that the real part of dielectric constant is independent of frequency and the loss factor, ϵ''/ϵ', due to dielectric dispersion, is very small and linearly increasing with frequency. Also at frequencies well above the resonant frequency approximate expressions for the real and imaginary parts of the dielectric constant can be obtained and are given by:

$$\epsilon' \simeq \epsilon_0\left(1 - \frac{\omega_\text{p}^2}{\omega^2}\right)$$

$$\epsilon'' \simeq \epsilon_0\,\frac{\gamma\omega_\text{p}^2}{\omega^3}$$

It is interesting to note that in the limit as $\omega \to \infty$, the real part of permittivity approaches that of free space and the imaginary part vanishes.

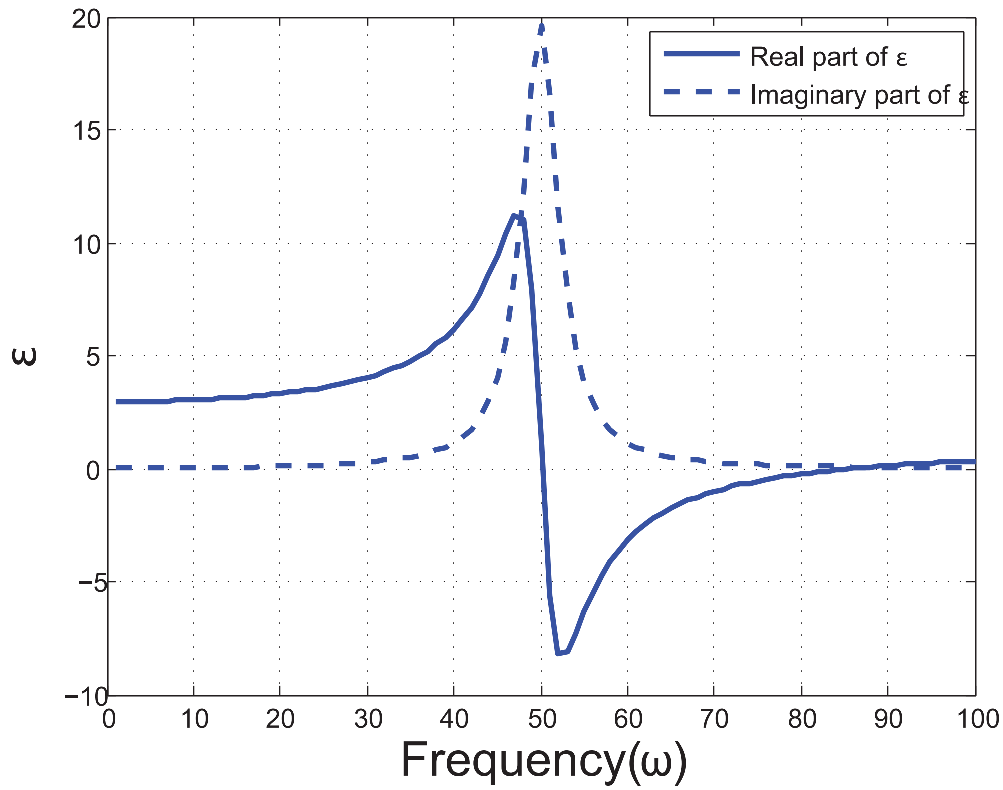

Figure 3: The real and imaginary parts of dielectric constant of a material predicted by Lorentz model ($\omega_0 = 50,\ \omega_\mathrm{p} = 70,\ \gamma = 5$).

The Drude Dielectric Model for Metals

Conductors are referred to materials with very high conductivity ($\sigma \gg 1$). As will be shown later the material conductivity can be expressed in terms of frequency dependent imaginary part of dielectric constant given by

$$\epsilon''_\mathrm{cond} = \frac{\sigma}{\omega}$$

Hence a standard dielectric model for metals with finite conductivity is expressed as

$$\epsilon_\mathrm{metal} = \epsilon_0 \left(1 + i\,\frac{\sigma}{\omega\epsilon_0}\right)$$

This model is based on Ohm's law and is valid at low frequencies ($f < 100$ GHz). At higher frequencies, a better approach to describe the spectral behavior of conductors is the Drude model. This model can be derived from the Lorentz dielectric model explained in the previous section. For good conductors there are a large number of electrons at the top of the energy distribution that can easily be excited and moved around within the conduction band. These electrons are essentially free electrons and not bound to specific ions in the material lattice. As a result the spring constant S in the equivalent mechanical model shown in Fig. 16 can be set to zero. Essentially by setting $\omega_0 = 0$ in (16), the Drude dielectric model for conductors is given by

$$\epsilon = \epsilon_0 \left[1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2 + i\gamma\omega} \right]$$

The explicit expressions for the real and imaginary parts of the dielectric constant are given by

$$\epsilon' = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2 + \gamma^2}$$
$$\epsilon'' = \frac{\omega_{\mathrm{p}}^2 \gamma}{\omega(\omega^2 + \gamma^2)}$$

Kramers-Krönig's Relation

The nature of complex permittivity of materials given by (12) is not arbitrary. It can be shown that the real and imaginary parts of the complex permittivity are related to each other. In addition to the causality condition that led to derivation (12), we further postulate that the system of charges in the material is unconditionally stable. That is, the impulse response $\chi_{\mathrm{e}}(t) \to 0$ as $t \to \infty$. This will guarantee that the Fourier transform $\chi_{\mathrm{e}}(\omega)$ has no poles in the upper half-plane of the complex ω-plane where $\mathrm{Im}(\omega) \geq 0$. Equation (12) also implies that $\chi_{\mathrm{e}}(\omega) \to 0$ as $\omega \to \infty$ in the upper half-plane.

Now consider the contour integral

$$I = \oint_C \frac{\chi_{\mathrm{e}}(\omega')}{\omega' - \omega} \, d\omega' \tag{21}$$

where the closed contour C is composed of the real axis and a semicircle in the upper half-plane whose radius approaches infinity as shown in Fig. 4. Since there are no poles

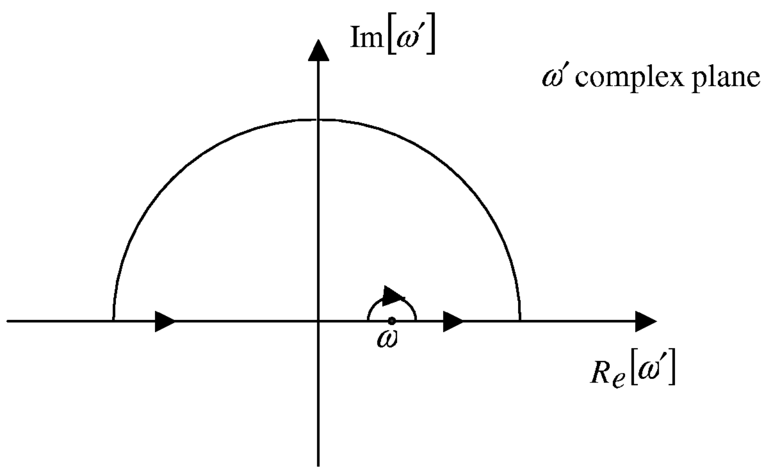

Figure 4: Complex ω'-plane and closed contour C.

and branch cuts associated with the integrand of (21) within the contour C, according to Cauchy's theorem $I = 0$. Also according to Jordan's lemma the integral over the semicircle of infinite radius vanishes. Denoting the principal value of the integral as the integral over the real axis except at $\omega = \omega'$ and evaluating one-half the residue at $\omega = \omega'$ it can be shown that

$$-i\pi\chi_{\rm e}(\omega) + \fint_{-\infty}^{+\infty} \frac{\chi_{\rm e}(\omega')}{\omega' - \omega}\, d\omega' = 0 \tag{22}$$

Representing the real and imaginary parts of the complex susceptibility as $\chi_{\rm e}(\omega) = \chi'_{\rm e}(\omega) + i\chi''_{\rm e}(\omega)$, it can easily be shown that

$$\chi'_{\rm e}(\omega) = \frac{1}{\pi}\fint_{-\infty}^{+\infty} \frac{\chi''_{\rm e}(\omega')}{\omega' - \omega}\, d\omega' \tag{23}$$

$$\chi''_{\rm e}(\omega) = -\frac{1}{\pi}\fint_{-\infty}^{+\infty} \frac{\chi'_{\rm e}(\omega')}{\omega' - \omega}\, d\omega' \tag{24}$$

Clearly (23) and (24) show that the real and imaginary parts of the susceptibility function are related to each other through a pair of integral transforms known as Kramers-Krönig Relations. Using (12) a similar relationship between the real and imaginary parts of the permittivity ($\epsilon'_{\rm r}(\omega)$) can be obtained and are given by

$$\epsilon'_{\rm r}(\omega) - \epsilon_{\rm r\infty} = \frac{1}{\pi}\fint_{-\infty}^{+\infty} \frac{\epsilon''_{\rm r}(\omega')}{\omega' - \omega}\, d\omega'$$

$$\epsilon''_{\rm r}(\omega) = -\frac{1}{\pi}\fint_{-\infty}^{+\infty} \frac{\epsilon'_{\rm r}(\omega') - \epsilon_{\infty}}{\omega' - \omega}\, d\omega'$$

where $\epsilon_{\rm r\infty}$ is the value of permittivity at infinite frequency. Strictly speaking $\epsilon_{\rm r\infty} = 1$ according to (12).