Chapter 12

Frequency-Hopped Spread-Spectrum

In this chapter we discuss frequency-hopped spread-spectrum. We first describe the antijam capability, then the multiple-access capability and finally the fading resistance.

Frequency-hopped spread-spectrum works by pseudorandomly changing the center frequency of the carrier over a set of frequencies. The sequence of frequencies used is called the frequency hopping pattern. In a jamming environment this can force the jammer to spread his power over a very wide bandwidth in order to guarantee that the transmitted signal is disrupted (to some extent). When the jammer spreads his signal over the whole bandwidth the amount of power in each frequency slot is a small fraction of the total power. Thus the effectiveness of a wideband jammer is reduced in proportion to the bandwidth over which the signal is spread. If this the spreading is large enough the jammer is not effective at disrupting communications. If the jammer only jams a fraction of the band but with high power in certain slots then the performance can be severely degraded (as opposed to a wideband jammer with the same total power). However, in this case a proper error control code and decoding algorithm that corrects errors in slots jammed can change the optimal jamming strategy from narrowband to broadband and thus regain the advantage of spreading the spectrum.

For multiple-access different users have different frequency hopping patterns and different users will collide occasionally. These collisions can be handled with the use of appropriate error control coding. A code with a very low rate is needed for a large number of interferers while for a small number of interfers a large rate code should be employed. To maximize the information throughput per unit bandwidth an optimal code rate and number of users can be found.

For fading channels provided the frequency separation between slots is larger then the coherence bandwidth different frequencies will fade independently. If the bandwidth within a slot is small compared to the coherence bandwidth then the fading will be nonselective within a hop. An error control code will be able to correct errors from a badly faded hop. It is interesting to note that the (uncoded) performance in a fading environment is actually worse than the performance in a jamming environment. In both cases the performance degradation (without coding) is on the order of 30-40dB compared to additive white Gaussian noise. This can be reduced somewhat in the partial-band jamming case by spreading over a very large bandwidth. However, with the proper combination of coding and spreading the loss can be completely recovered without such large bandwidth expansion.

The rest of the chapter is organized as follows. First we discuss how frequency hopping works. Next we discuss the performance in channels with jamming. We next discuss the performance in channels with multiple-access interference.

1. Introduction

The block diagram of a frequency-hopped spread-spectrum systems is shown below. The information to be transmitted is first encoded using an error control code. This information is modulated using some intermediate frequency. Last, the signal is mixed up to a carrier frequency that is pseudo-randomly changing.

The receiver operates as follows. The received signal is mixed down to some (constant) intermediate frequency before being demodulated. The output of the demodulator is then processed by the decoder to recover the transmitted information. In addition the decoder may know some information about the reliability of each symbol. This

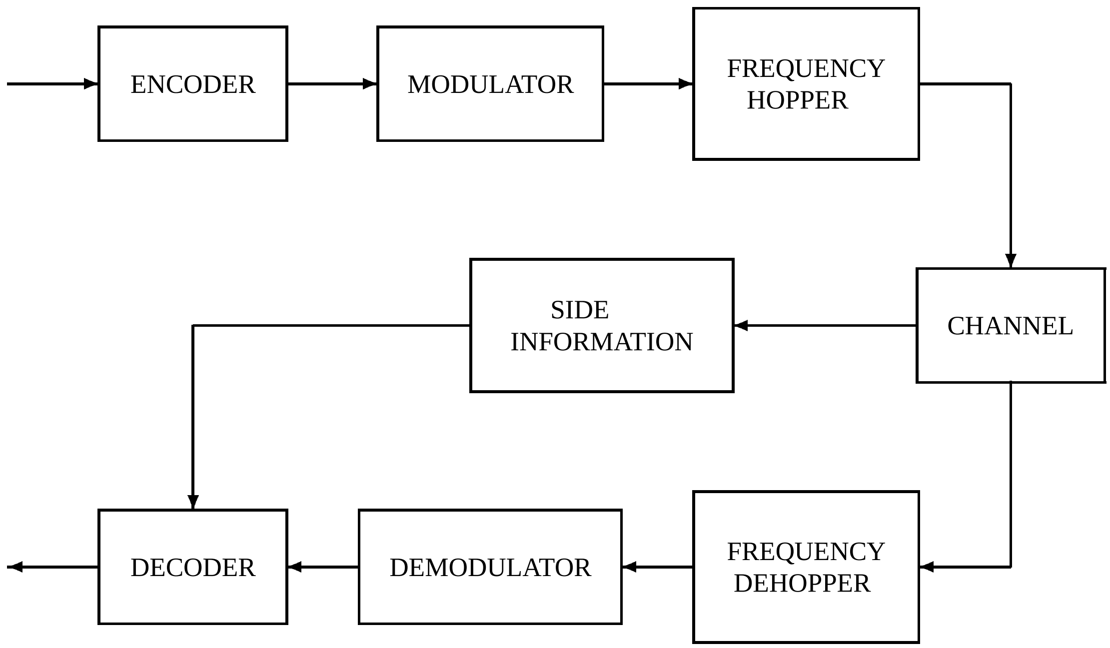


Figure 12.1: Block Diagram of Frequency Hopped Spread-Spectrum System.

Figure 12.2: Frequency Hopping Pattern for User 1

information (called side-information) may be the fade level in a particular hop, the jamming level in a hop, or whether there was a hit in a particular hop. This information (when used properly) can significantly improve the performance of the decoding.

Several frequency hopping patterns for different users are shown below. Also different jamming strategies are shown.

2. Frequency Hopping with Jamming

Worst-Case Interference Rejection

Assume we are using binary frequency shift keying (BFSK) with frequency-hopped spread-spectrum. The demodulation is noncoherent matched filtering (to each of the two dehopped frequencies) as shown below.

The receiver will make an error if symbol 1 is transmitted but $Y_{-1} > Y_1$. The error probability of such a modulation with energy E_b, when demodulated noncoherently in the presence of additive white Gaussian noise with (two sided) power spectral density $N_0/2$ is

$$P_e = \frac{1}{2}\exp\{-\frac{E_b}{2N_0}\}.$$

Consider now several scenarios of jamming and spreading.

Figure 12.3: Frequency Hopping Pattern for User 2

Figure 12.4: Frequency Hopping Pattern for User 3

Figure 12.5: Frequency Hopping Pattern for User 4

Uncoded, Unspread Signal, Unspread Jammer

The jammer has power J and the communicator (transmitter) has power P. The communicator's signal occupies a bandwidth $W = R_b$ where R_b is the rate of transmission (information bits per second). Let $E_b = P/R_b$ be the energy available per information bit transmitted. If the jammer concentrates his (Gaussian) signal over the signal bandwidth W, then the (one-sided) spectral density is $N_0 = J/W$ and the performance (error probability) of a modulation scheme will be a function of

$$E_b/N_0 = \frac{P}{J}\frac{W}{R_b} = \frac{P}{J}$$

and

$$P_e = \frac{1}{2}\exp\{-\frac{E_b}{2N_0}\} = \frac{1}{2}\exp\{-\frac{P}{2J}\}.$$

Uncoded, Spread Signal, Unintelligently Spread Jammer

Now the communicator spreads his signal over a bandwidth W. The rate of transmission (information bits per second) is R_b. Let $q = W/R_b$ be the spreading factor (also called the processing gain). If the jammer spreads his (Gaussian) signal over the whole bandwidth, then the (one-sided) spectral density is $N_0 = J/W$ and the performance (error probability) of a modulation scheme will be a function of

$$\frac{E_b}{N_0} = \frac{P}{J}\frac{W}{R_b} = \frac{P}{J}q$$

and

$$P_e = \frac{1}{2}\exp\{-\frac{E_b}{2N_0}\} = \frac{1}{2}\exp\{-\frac{Pq}{2J}\}.$$

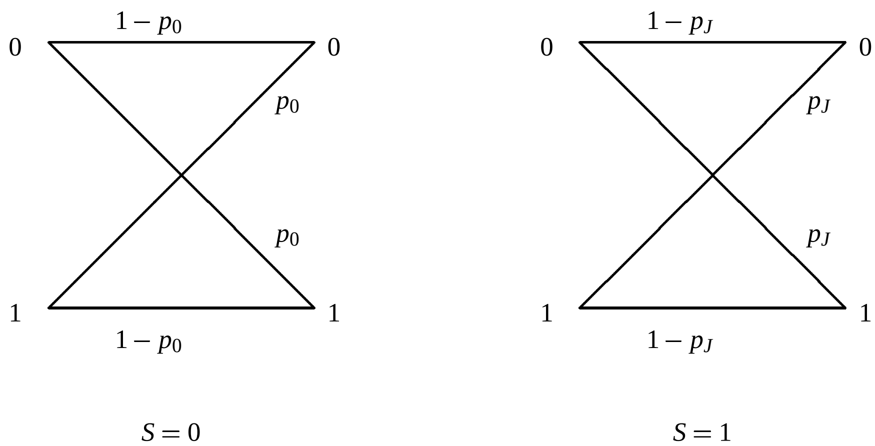
Thus the communicator has gained $10\log_{10} q$ dB in signal-to- noise ratio over an unspread system. The functional form of the error probability is same as an unspread system.

Now the jammer spreads his (Gaussian) signal over *a fraction*, ρ, of the whole bandwidth. The remaining fraction is noise free. Then the (one-sided) spectral density is over the fraction jammed is $N_0 = J/(W\rho)$. However, only a fraction ρ of the bits transmitted are jammed. In the absence of coding we make a decision on each symbol transmitted. The model is that of a two state channel as shown below. In one state the channel has no jamming interference while in the other state the jammer is present.

Figure 12.6: Slow Frequency Hopping Pattern

Figure 12.7: Full Band Jammer

Figure 12.8: Partial Band Jammer



The transition probability p_0 is the error probability in the absence of jamming and p_j is the error probability with jamming.

$$
\begin{aligned}
P_e &= P\{S=0\}p_0 + P\{S=1\}p_J \\
&= (1-\rho)p_0 + \rho p_J \\
&= (1-\rho)0 + \rho\frac{1}{2}\exp\{-\frac{P/R_b}{2(J/W\rho)}\} \\
&= \frac{\rho}{2}\exp\{-\frac{Pq\rho}{2J}\} = \frac{\rho}{2}\exp\{-\frac{E_b\rho}{2N_J}\}
\end{aligned}
$$

where we have assumed that the error probability for the unjammed channel is 0.

The intelligent jammer chooses ρ to maximize the error probability.

$$
P_e = \max_{0\le\rho\le 1}\frac{\rho}{2}\exp\{-\frac{Pq\rho}{2J}\} = \begin{cases} \frac{1}{2}\exp\{-\frac{Pq}{2J}\}, & Pq/J \le 2 \\ \frac{e^{-1}}{Pq/J}, & Pq/J \ge 2. \end{cases}
$$

It is no longer simple to make a comparison between the above systems just in terms of the signal-to-noise ratio since the error probability has a different functional form. However, it is easy to plot these as a function of P/J for a particular q.

For any spread system let $N_J = J/W$. The ratio $\frac{Pq}{J}$ is just $\frac{E}{N_J}$. The error probability for spread spectrum systems is usually plotted as a function of $\frac{E}{N_J}$.

Important issues in coding for a spread-spectrum system in the presence of a partial-band jammer.

- Hard Decisions/Soft Decisions
- Side information/No side information
- Burst Noise/Memoryless Channel

Figure 12.9: Partial Band Jammer

Figure 12.10: Partial Band Jammer

Figure 12.11: Partial Band Jammer

It is almost always sufficient when analyzing a coded system to consider the performance of a repetition code of length L. The performance of other codes can usually be obtained from the performance of a repetition code. So assume the two possible codewords transmitted are

$$(1,\ 1,\ 1,\cdots,\ 1) \text{ and } (-1,\ -1,\ -1,\cdots,\ -1)$$

and that the transmitted codeword is

$$(1,\ 1,\ 1,\cdots,\ 1).$$

These codewords are of length L. The receiver output consists of L pairs of random variables.

$$\begin{array}{cccc} Y_{-1}(1) & Y_{-1}(2) & \cdots & Y_{-1}(L) \\ Y_1(1) & Y_1(2) & \cdots & Y_1(L) \end{array}$$

where Y_{-1} is the output of the filter matched to the frequency for symbol -1 and Y_1 is the output of the filter matched to the frequency for symbol 1.

Performance measures

- Bit error rate for specific codes.
- Capacity
- $\rho^*(p_b)$ minimum fraction of band that must be jammed for the resulting error probability to be greater than p_b independent of the jammer's power or distribution.

Channel Models

Input: $X \in \{-1,1\}$
Output $\mathbf{Y} = (Y_{-1},Y_1) \in (R^+)^2$
State: $S \in \{0,1\}$
$S = 0 \Rightarrow$ No interference
$S = 1 \Rightarrow$ Interference
$P\{S = 0\} = 1-\rho$
$P\{S = 1\} = \rho$
X is the input to a BFSK modulator.
Y_i, $i = -1,1$ is the output of the (noncoherent) filter matched to the i-th signal.
Transition Probabilities

$$p(y_i|X = l,S) = \begin{cases} \frac{1}{2\sigma^2}\exp[-(\frac{y_i+\Lambda}{2\sigma^2})]I_0(\sqrt{y_i\Lambda}/\sigma^2), & i = l \\ \frac{1}{2\sigma^2}\exp[-(\frac{y_i}{2\sigma^2})], & i \neq l \end{cases}$$

where

$$\sigma^2 = SN_J/2\rho + N_0/2$$

and

$$\Lambda = E$$

$$p(\mathbf{y}|X = l,S) = \prod_{i=0}^{1} p(y_i|X = l,S)$$

Furthermore we will assume the channel is memoryless.

Figure 12.12: Partial Band Jammer

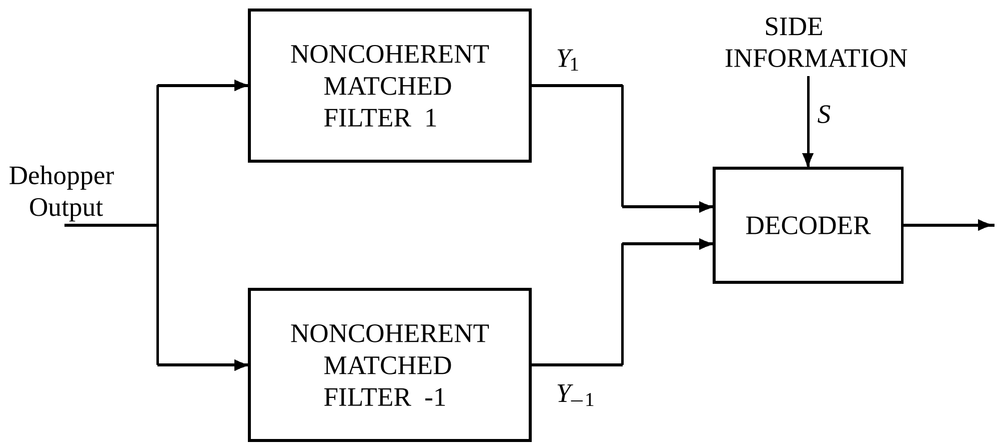


Figure 12.13: Demodulation/Decoding for Frequency Hopping

Figure 12.14: Error Probability without Spreading or Coding

Decoding for Hard Decisions without Side Information.

This decoder makes a decision on each of the symbols of a codeword and then does majority logic combining. For example

$$\begin{array}{ccccccccccc} (Y_{-1}(1) & Y_{-1}(2) & \cdots & Y_{-1}(L)) & = & (0.2, & 1.4, & 0.6, & 0.3, & 1.7) \\ (Y_1(1) & Y_1(2) & \cdots & Y_1(L)) & = & (0.7, & 0.4, & 0.8, & 0.1, & 1.2) \end{array}$$

The decoder would decide the following sequence of transmitted symbols

$$(1, -1, 1, -1, -1).$$

Since there are more -1's than 1's the decoder would decide the codeword transmitted was $(-1, -1, -1, -1, -1)$.

The performance is characterized (independent of the code being used but assuming majority logic decoding) by

$$D = \max_{0 \le \rho \le 1} 2\sqrt{\bar{p}(1-\bar{p})}$$

where

$$\bar{p} = \rho p_J + (1-\rho) p_0.$$

Decoding for Hard Decisions with Side information.

When the decoder has side information available the decision rule is to first ignore all positions that have been interfered with and then decode the symbols that have not been interfered with. In the case where all symbols have been interfered with the decoder does majority logic decoding. For example

$$\begin{array}{ccccccccccc} (Y_{-1}(1) & Y_{-1}(2) & \cdots & Y_{-1}(L)) & = & (0.2, & 1.4, & 0.6, & 0.3, & 1.7) \\ (Y_1(1) & Y_1(2) & \cdots & Y_1(L)) & = & (0.7, & 0.4, & 0.8, & 0.1, & 1.2) \\ (S(1) & S(2) & \cdots & S(L)\ ) & = & (1 & 0 & 1 & 0 & 0) \end{array}$$

Figure 12.15: Error Probability without Spreading or Coding

Figure 12.16: Error Probability with Spreading and Worst-Case Partial-Band Interference

Figure 12.17: Error Probability with Spreading and Worst-Case Partial-Band Interference

The decoder would decide the following sequence of transmitted symbols

$$(1,-1,1,-1,-1).$$

It would ignore the first and third positions and then decide that the sequence

$$(-1,-1,-1,-1,-1).$$

was the transmitted sequence.

The performance is characterized (independent of the code being used) by

$$D = \max_{0\le\rho\le 1} \rho 2\sqrt{p_J(1-p_J)} + (1-\rho)2\sqrt{p_0(1-p_0)}$$

Decoding for Soft Decisions with Side Information.

If any position of the codeword is demodulated without any interference then the decoder can make the right choice for which codeword was transmitted. Only if all positions are interfered with will the decoder possible make an error. If this occurs the decoding is the same as for a white Gaussian noise channel. (Optimum Decoding for white Gaussian noise and noncoherent demodulation is very complicated involving Bessel functions etc.)

For noncoherent demodulation with optimum combining and assuming negligible background noise

$$\begin{aligned} D &= \max_{0\le\rho\le 1} \rho\Big[\int_0^\infty u e^{-u^2/2} I_0^{1/2}(u\sqrt{2E\rho/N_J})du\Big]^2 \\ &= \begin{cases} [\int_0^\infty u e^{-u^2/2} I_0^{1/2}(u\sqrt{2E/N_J})du]^2, & E/N_J < 2.871 \\ \frac{1.424}{E/N_J}, & E/N_J \ge 2.871 \end{cases} \end{aligned}$$

Decoding for Soft Decisions (Square-Law Combining) with Side Information.

For noncoherent demodulation with square-law combining is suboptimum for AWGN but fairly easy to implement.

$$\sum_{i=1}^{L}(1-S_i)Y_1(i) \underset{\text{dec } -1}{\overset{\text{dec } 1}{\gtrless}} \sum_{i=1}^{L}(1-S_i)Y_{-1}(i)$$

where the sum includes only the terms not jammed The performance of square law combining is determined from

$$\begin{aligned} D &= \max_{0\le\rho\le 1}\min_{0\le\lambda\le 1}\left[\frac{\rho}{1-\lambda^2}\exp\{-\lambda E/N_J/(1+\lambda)\}\right]. \\ &= \begin{cases} \min_{0\le\lambda\le 1}[\frac{1}{1-\lambda^2}\exp\{-\lambda E/N_J/(1+\lambda)\}] & E/N_J < 3. \\ \frac{4e^{-1}}{E/N_J} & E/N_J \ge 3 \end{cases} \end{aligned}$$

As with optimum combining if any symbol is received without jamming then the decoder will not make an error. Only if all symbols (in a codeword) are jammed will the decoder possibly make an error. If all symbols are jammed then the decoder does square-law combining.

Notice that for reasonable signal-to-jamming noise ratios the loss in performance by using square-law combining is $10\log_{10}(4e^{-1}/1.424)$ =0.14dB. Since square-law combining is much easier to implement than optimum, there seems no reason for considering optimum combining.

Decoding for Soft Decisions without Side Information.

Consider square-law combining without side information

$$\begin{aligned} D &= \max_{0\leq\rho\leq 1}\min_{0\leq\lambda\leq 1}\left[\frac{\rho}{1-\lambda^2}\exp\{-\lambda E/N_J/(1+\lambda)\} + (1-\rho)\exp\{-\lambda E/N_J\}\right] \\ &= 1 \end{aligned}$$

This metric performs extremely poorly in the presence of worst-case jamming. The optimal jamming strategy is to jam a very small fraction of the band, i.e $\rho \approx 0$. Essentially the jammer would only jam one symbol of a codeword with enough power to cause an error.

Decoding for Soft Decisions without Side Information.

Consider the following normalization with frequency shift keying and noncoherent demodulation. The receiver normalizes the outputs of each of the noncoherent matched filters by the sum of the outputs. That is, if Y_i i=0,1 are the outputs then the receiver computes

$$Z_i = \frac{Y_i}{Y_1 + Y_{-1}}, \quad i = -1, 1$$

The decoder then processes the Z_i in a linear fashion. Note that $0 \leq Z_i \leq 1$.

$$D = \max_{0\leq\rho\leq 1}\min_{\lambda\geq 0}\left[\frac{\rho}{(2\lambda-\gamma_J)^2}\left(2\lambda e^{-\gamma_J+\lambda} + (\gamma_J^2 - 2\lambda - 2\lambda\gamma_J)e^{-\lambda}\right)\right]$$

where

$$\gamma_J = \rho E/N_J.$$

This processing performs very well for the nonbinary case and as M becomes large approaches that of perfect side information. (For $M = 2$ it is a soft decision version of Viterbi Ratio Threshold). For large E/N_J it yields

$$D \approx \frac{0.893}{(E/N_J)^{0.6144}}$$

Below is a summary of the different capacities of different channels when there is a interferer with unknown parameter ρ.

- Soft Decisions
  - No Side Information

$$C = \max_X \min_{0\leq\rho\leq 1} I(X;Y).$$

  - Side Information

$$C = \max_X \min_{0\leq\rho\leq 1} I(X;Y,S).$$

- Hard Decisions

$$\tilde{Y} = \arg\max\{Y_{-1}, Y_1\}.$$

Figure 12.18: Comparison of D for different decoding techniques.

Figure 12.19: Bit error probability for repetition codes of length 1,3,5,7 on two state channels with and without side information.

- No Side Information

$$C = \max_X \min_{0 \le \rho \le 1} I(X;\tilde{Y}).$$

- Side Information

$$C = \max_X \min_{0 \le \rho \le 1} I(X;\tilde{Y},S).$$

- Soft Decisions
  - Side Information

$$C = \begin{cases} C(E/N_J), & E/N_J \le 2.41 \\ 1 - \frac{1.17}{E/N_J}, & E/N_J \ge 2.41. \end{cases}$$

where

$$C(x) = e^{(-x^2/2)} \int_0^\infty \int_0^\infty y_0 y_1 e^{-(y_0^2+y_1^2)} f(x,y_0,y_1) dy_0 dy_1.$$

and

$$f(x,y_0,y_1) = I_0(2xy_0) \log_2 \left\{ \frac{2I_0(2xy_0)}{I_0(2xy_0) + I_0(2xy_1)} \right\}.$$

- Hard Decisions
  - No Side Information

$$C = C(P_e(E/N_J))$$

where

$$P_e(E/N_J) = \begin{cases} \frac{1}{2}\exp\{-\frac{E}{2N_J}\}, & E/N_J \le 2 \\ \frac{e^{-1}}{E/N_J}, & E/N_J \ge 2. \end{cases}$$

  - Side Information

$$C = \begin{cases} C(P_e(E/N_J)), & E/N_J \le 3.02 \\ 1 - \frac{1.51}{E/N_J}, & E/N_J \ge 3.02. \end{cases}$$

where

$$P_e(E/N_J) = \frac{1}{2} e^{-E/2N_J}$$

and

$$C(p) = 1 - H_2(p) = 1 + p\log(p) + (1-p)\log(1-p).$$

Coding Theorem for Compound Channel

Figure 12.20: Bit error probability for Reed-Solomon codes and upper bound on bit error probability for convolutional codes on two state channels without side information.

Figure 12.21: Bit error probability for Reed-Solomon codes and upper bound on bit error probability for convolutional codes on two state channels with side information.

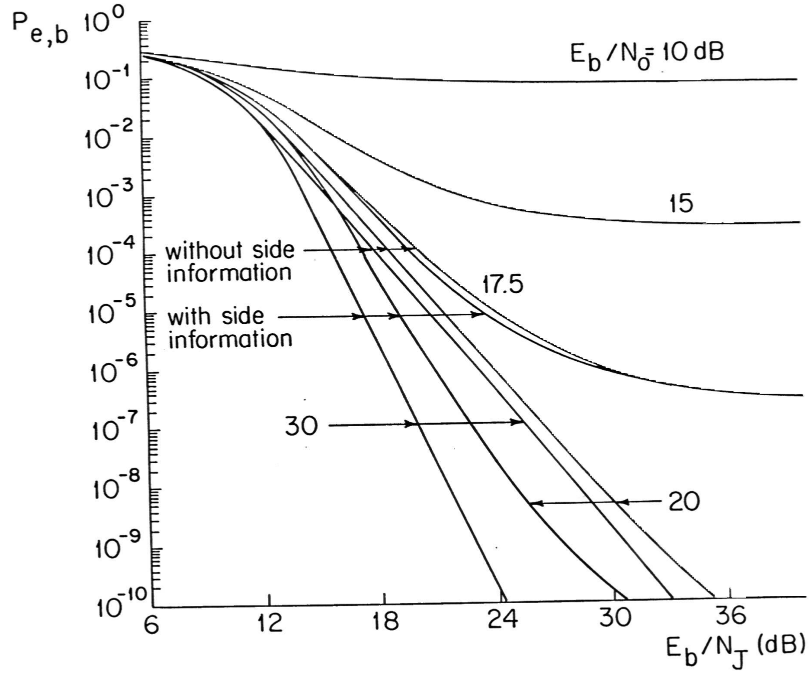


Figure 12.22: Bit error probability for length 7 repetition codes on worst-case partial band jamming channel with binary FSK.

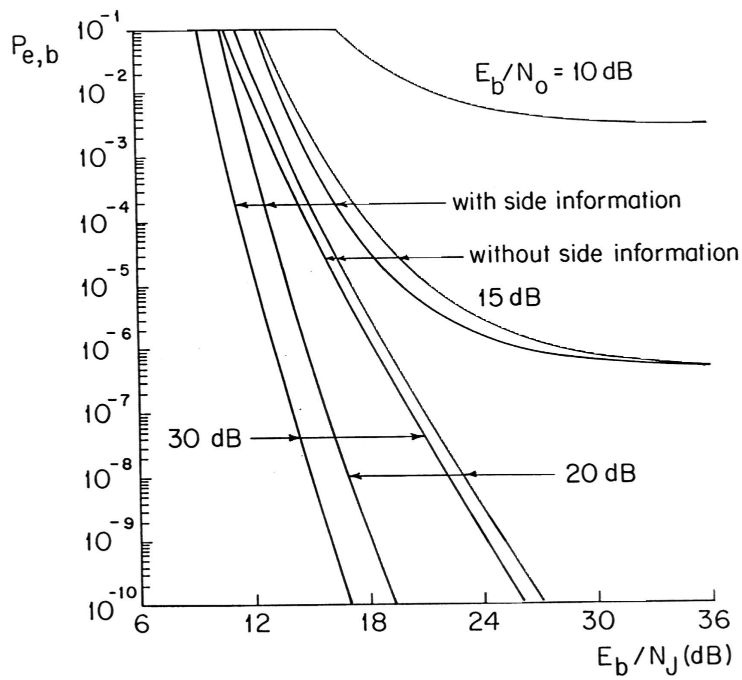


Figure 12.23: Upper bound on bit error probability for constraint length 7 convolutional code with and without side information on worst-case partial-band jamming channel with binary FSK.

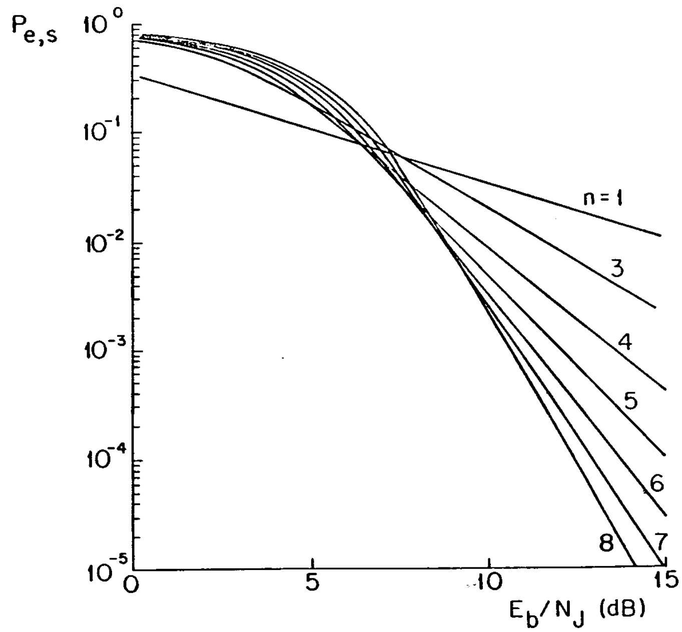


Figure 12.24: Symbol error probability for repetition codes on channels with hard decisions and no side information on worst-case partial-band jamming channel and 32-ary FSK ($E_b/N_0 = \infty$).

Figure 12.25: Symbol error probability for repetition codes on channels with hard decisions with side information on worst-case partial-band jamming channel and 32-ary FSK ($E_b/N_0 = \infty$).

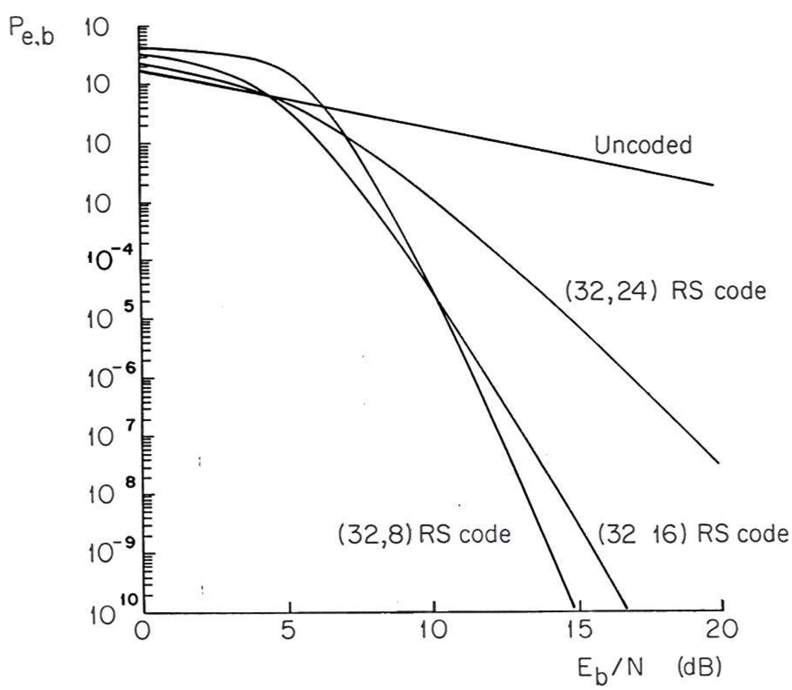


Figure 12.26: Bit error probability for Reed-Solomon codes and 32-ary FSK on channel without side information and worst-case partial-band jamming channel ($E_b/N_0 = \infty$).

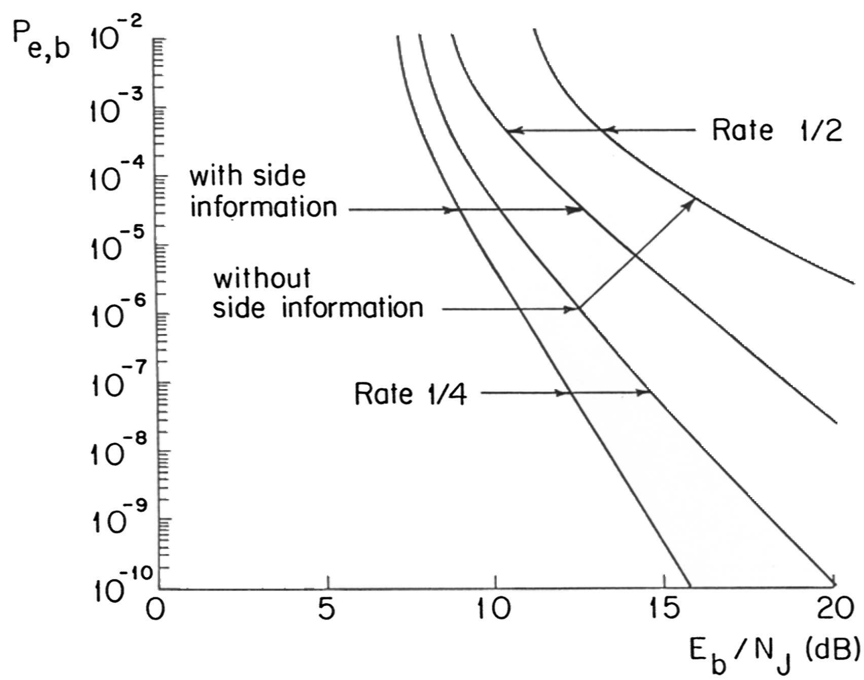


Figure 12.27: Upper bound on bit error probability for Dual-5 convolutional codes and 32-ary FSK on channel with worst-case partial-band jamming channel ($E_b/N_0 = \infty$).

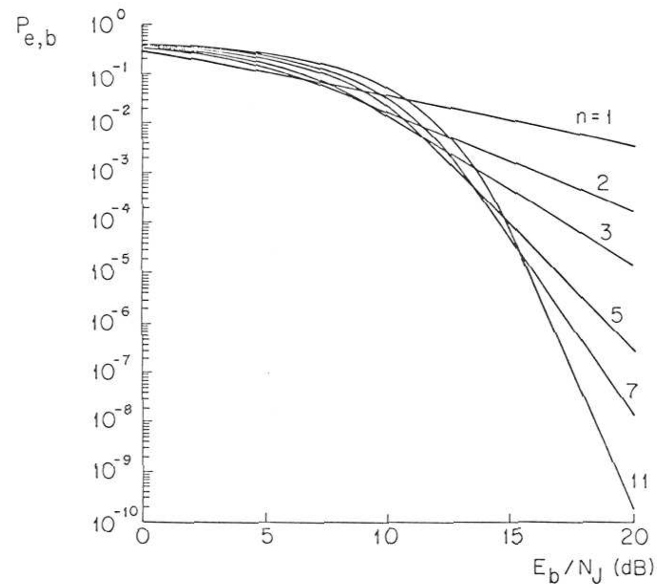


Figure 12.28: Bit error probability for repetition codes and binary FSK on channel with worst-case partial-band jamming channel and soft decision decoding with side information ($E_b/N_0 = \infty$).

Figure 12.29: Bit error probability for rate 1/2, constraint length 7 convolutional codes and binary FSK on channel with worst-case partial-band jamming ($E_b/N_0 = \infty$).

Figure 12.30: Symbol error probability for repetition codes and 32-ary FSK on channel with side information and soft decision and worst-case partial-band jamming ($E_b/N_0 = \infty$).

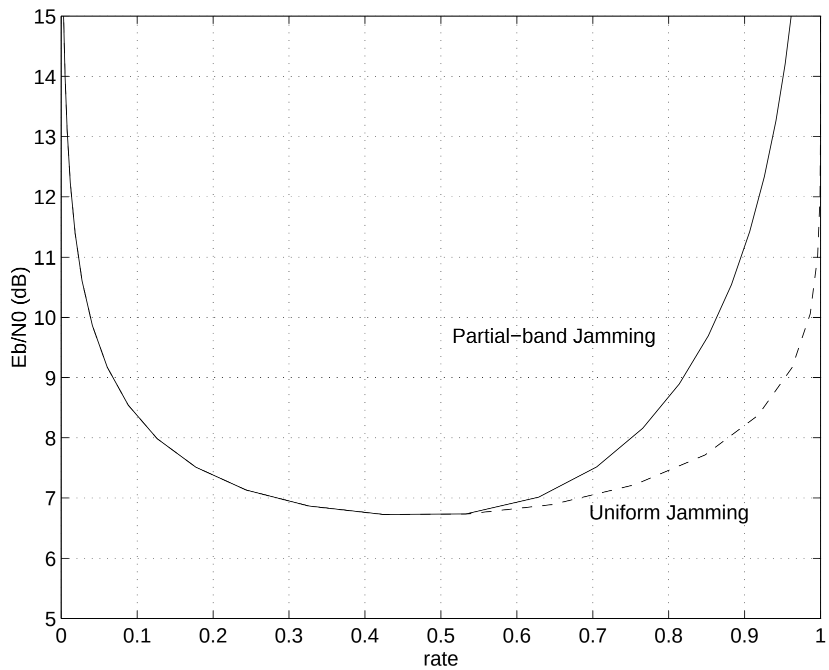


Figure 12.31: E_b/N_J needed to achieve capacity for binary FSK with soft decisions and side information available

There exist codes of rate r (information bits per channel use) and a decoding algorithm such that reliable communication (arbitrarily small error probability) is possible if

$$r < C(E/N_J)$$

where E= energy per channel use. Note that the decoder does not know which value of ρ is going to be used apriori. It must be able to communicate at rate r (below the capacity) no matter which value of ρ is being used by the communicator. The value of ρ does not change during transmission of a codeword.

Let E_b be the energy needed per information bit and r be the number of information bits per channel bit. Then

$$E_b = E/r$$

Interpretation:

$$r < C(E/N_J)$$
$$\Downarrow$$
$$E/N_J > C^{-1}(r)$$
$$\Downarrow$$
$$E_b/N_J > \frac{C^{-1}(r)}{r}$$

Thus $C^{-1}(r)/r$ is the minimum signal-to-noise ratio needed for reliable communication when codes of rate r are used.

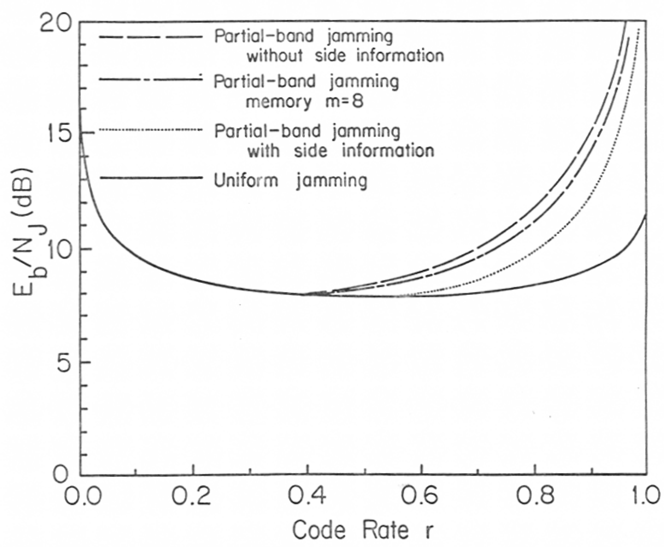


Figure 12.32: E_b/N_J needed to achieve capacity for binary FSK with hard decisions, with and without side information.

Figure 12.33: E_b/N_J needed to achieve capacity for 8-ary FSK with hard decisions.

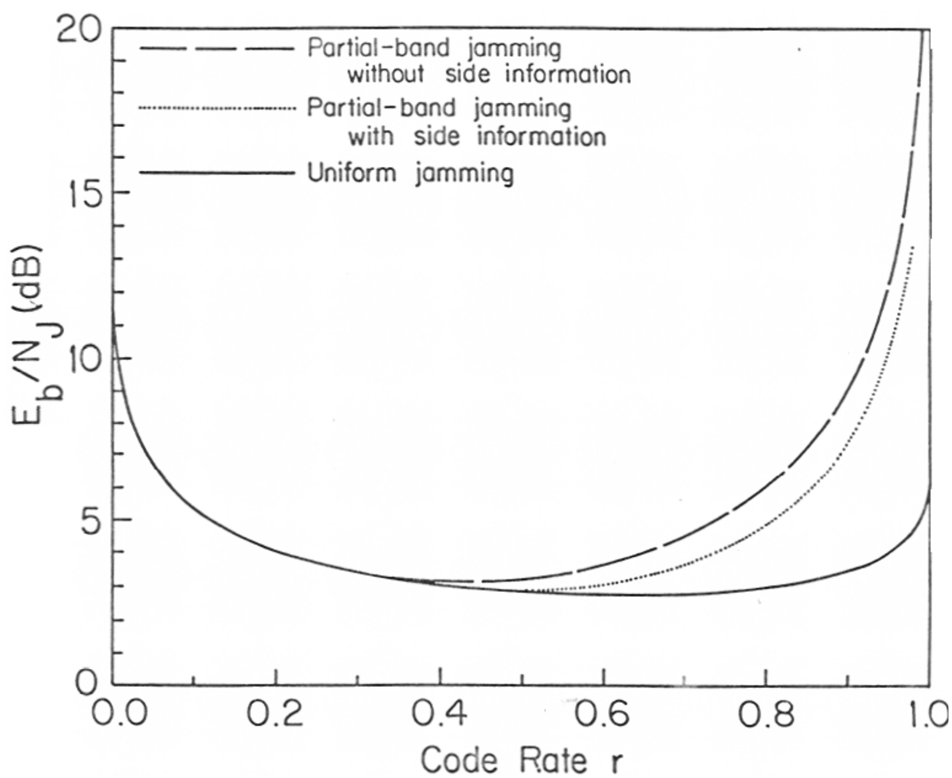


Figure 12.34: E_b/N_J needed to achieve capacity for 32-ary FSK with hard decisions.

Figure 12.35: E_b/N_J needed to achieve cutoff rate for binary FSK with soft decisions and side information available

Figure 12.36: E_b/N_J needed to achieve cutoff rate for binary FSK with hard decisions.

3. Multiple-Access Capability

In this section we consider a frequency-hopped spread-spectrum system used for multiple-access. The model for the system is shown below. There are K users who are simultaneously transmitting information. Each user is assigned a unique hopping pattern. However, because the users are not synchronous there is the chance that the signal of the two users could hop to the same frequency at the same time when received at a particular receiver. In this case we say a hit has occurred. In this section we consider two simple models (one symbol/hop) for a K user,

Figure 12.37: Block Diagram of Frequency-Hopped Multiple-Access System

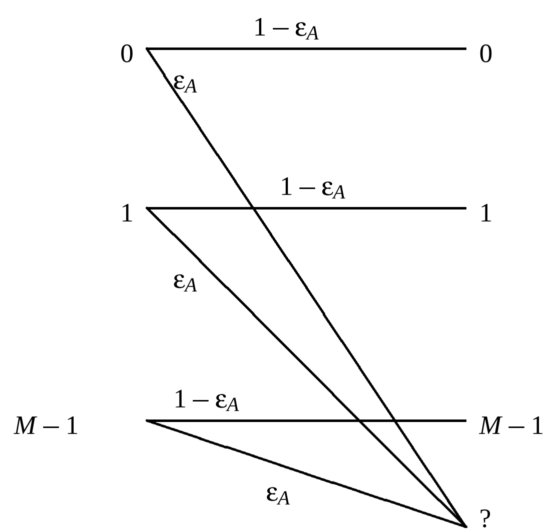
q frequency slot system with code rate r. In the first model the receiver knows when a particular symbol has been hit by another user and that symbol is erased. In the second model the receiver does not have knowledge of hits. When a hit occurs there is a much higher probability of error than in the absence of a hit. We do not distinguish between partial hits and full hits. The purpose is to illustrate the optimization. That is, for a given bandwidth and number of frequency slots there is an optimal number of users that should be transmitting with an optimal rate code to maximize the amount of information being reliably received.

The two models are shown below.

With Side Information

Erase symbols that are hit by other users.

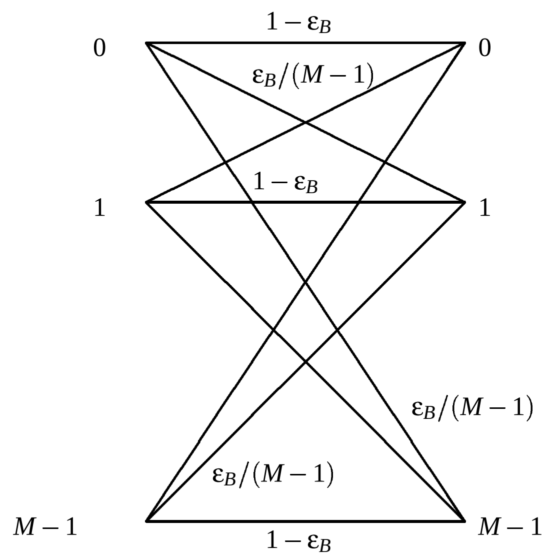
$$\varepsilon_A = P\{\text{erase}\} = 1-(1-p_h)^{K-1}, \qquad p_h = \begin{cases} \frac{2}{q}-\frac{1}{q^2}, & \text{asynchronous} \\ \frac{1}{q}, & \text{synchronous.} \end{cases}$$



Without Side Information

Each symbol hit is in error with probability ε.

$$\varepsilon_B = P\{\text{error}\} = \varepsilon[1-(1-p_h)^{K-1}]$$



We assume that the output of the channel does not depend on which information symbols the interfering users are transmitting. The code we consider is a Reed-Solomon code. An (n,k) Reed-Solomon code can correct up to $n-k$ erasures or $\lfloor (n-k)/2 \rfloor$ errors. For example the $n=32$, $k=16$ code can correct 16 erasures and 8 errors and has rate 1/2. The NASA standard (255,223) code can correct 32 erasures or 16 errors and has rate 0.875.

Probability of Decoder Error The codeword error probability of a Reed-Solomon code can be easily determined by realizing the number of erasures (errors) follows a binomial distribution. Summing this over the number of erasures greater than the erasure correcting capability yields the probability of decoder error.

$$P_e(K,q,n,k) = \sum_{i=n-k+1}^{n} \binom{n}{i} \varepsilon_A^i (1-\varepsilon_A)^{n-i}$$

This formula is straight forward to evaluate but difficult to work with analytically. However, if we let $n \to \infty$ and $k \to \infty$ such that $k/n \to r$ then it is possible to get a much easier expression.

$$P_e(K,q,n,k) \to \begin{cases} 0, & r < 1-\varepsilon_A \\ 1/2, & r = 1-\varepsilon_A \\ 1, & r > 1-\varepsilon_A. \end{cases}$$

The interpretation is that for large length codes the number of errors is extremely close to the average number of errors. If the code has enough redundancy ($r < 1-\varepsilon_A$) then the code will be able to correct the average number of errors.

Reed-Solomon Codes with Error Correction

An (n,k) Reed-Solomon code can correct up to $\lfloor (n-k)/2 \rfloor$ errors.
Probability of Decoder Error

$$P_e(K,q,n,k) = \sum_{i=\lfloor (n-k)/2 \rfloor+1}^{n} \binom{n}{i} \varepsilon_B^i (1-\varepsilon_B)^{n-i}$$

As $n \to \infty$ and $k \to \infty$ such that $k/n \to r$

$$P_e(K,q,n,k) \to \begin{cases} 0, & r < 1-2\varepsilon_B \\ 1/2, & r = 1-2\varepsilon_B \\ 1, & r > 1-2\varepsilon_B. \end{cases}$$

Achievable Region (Large n, k)

Using the above asymptotic results we can get a region of code rates and number of users for which very low (near 0) error probability is possible.
With Side Information

$$\begin{aligned} r &< 1 - \varepsilon_A \\ r &< (1 - p_h)^{K-1} \\ K &< 1 + \frac{\ln(r)}{\ln(1 - p_h)} \end{aligned}$$

For large K, q ($K/q \to \lambda$)

$$\begin{aligned} r &< e^{-c\lambda} \\ \lambda &< \frac{-\ln(r)}{c} \end{aligned}$$

where

$$c = \begin{cases} 1 & \text{synchronous} \\ 2 & \text{asynchronous} \end{cases}$$

Without Side Information

$$\begin{aligned} r &< 1 - 2\varepsilon_B \\ r &< 2(1 - p_h)^{K-1} - 1 \\ K &< 1 + \frac{\ln(1 + r)/2}{\ln(1 - p_h)} \end{aligned}$$

For large K, q ($K/q \to \lambda$)

$$\begin{aligned} r &< 2e^{-c\lambda} - 1 \\ \lambda &< \frac{-\ln(1 + r)/2}{c} \end{aligned}$$

Reed-Solomon Codes
With Side Information

Throughput is a measure of the total traffic correctly transmitted and is defined as

$$w(r, K, q) = \frac{rK(1 - P_e(K, q, r))}{q}$$

It is clear that provided P_e is close to 0 that the throughput is a linear increasing function of r and K. In this case we can increase the throughput by increasing r and K. However, if K is too large and r is too large (the error correction capability is too small) then P_e increases. So there should be some optimal K and r that maximizes the throughput of the channel. Below we perform this optimization when the code length is very large. **Optimum Traffic**

$$\begin{aligned} K^*(r, q) &= 1 + \ln r/\ln(1 - p_h) \\ w^*(r, q) &= \frac{r}{q}(1 + \ln r/\ln(1 - p_h)) \end{aligned}$$

Optimum Code Rate

$$\begin{aligned} r^*(K, q) &= (1 - p_h)^{K-1} \\ w^*(r, q) &= \frac{K}{q}(1 - p_h)^{K-1} \end{aligned}$$

Optimum Code Rate and Traffic

$$K^*(q) = (\ln(1-p_h)^{-1})^{-1}$$

$$r^*(q) = e^{-1}(1-p_h)$$

$$w^*(q) = \frac{qe^{-1}}{(1-p_h)\ln(1-p_h)}$$

Asymptotic Analysis

$$\lambda^* = \lim_{q\to\infty} \frac{K^*(q)}{q} = \begin{cases} 1 & \text{synchronous} \\ 1/2 & \text{asynchronous} \end{cases}$$

$$r^* = \lim_{q\to\infty} r^*(q) = e^{-1}$$

$$w^* = \lim_{q\to\infty} w^*(q) = \begin{cases} e^{-1} & \text{synchronous} \\ e^{-1}/2 & \text{asynchronous} \end{cases}$$

Reed-Solomon Codes
With Out Side Information

Optimum Traffic

$$K^*(r,q) = 1 + \frac{\ln(1+r)/2}{\ln(1-p_h)}$$

$$w^*(r,q) = \frac{r}{q}(1 + \frac{\ln(1+r)/2}{\ln(1-p_h)})$$

Optimum Code Rate

$$r^*(K,q) = 2(1-p_h)^{K-1} - 1$$

$$w^*(r,q) = \frac{K}{q}(2(1-p_h)^{K-1} - 1)$$

Figure 12.38: Achievable Region of Rate and Traffic with Reed-Solomon Codes, with side information, $P_e = 10^{-2}$, $q = 100$.

Figure 12.39: Achievable Region of Rate and Traffic with Reed-Solomon Codes, without side information, $P_e = 10^{-2}$, $q = 100$.

Optimum Code Rate and Traffic

Optimum code rate and traffic can only be found numerically.

Asymptotic Analysis

$$\begin{aligned}
\lambda^* &= \lim_{q\to\infty} \frac{K^*(q)}{q} = \begin{cases} 0.1448 & \text{synchronous} \\ 0.0724 & \text{asynchronous} \end{cases} \\
r^* &= \lim_{q\to\infty} r^*(q) = 0.460 \\
w^* &= \lim_{q\to\infty} w^*(q) = \begin{cases} 0.314 & \text{synchronous} \\ 0.157 & \text{asynchronous} \end{cases}
\end{aligned}$$

Figure 12.40: Throughput for FHMA, (q=100) and length 32 Reed-Solomon Codes, (with side information).

Figure 12.41: Throughput for FHMA, (q=100) and length 256 Reed-Solomon Codes, (with side information).

Figure 12.42: Throughput for FHMA, (q=100) and length 32 Reed-Solomon Codes, (without side information).

Figure 12.43: Throughput for FHMA, (q=100) and length 256 Reed-Solomon Codes, (without side information).