Lecture Notes 2: Detection Theory

Goals:

- Optimum Detection in AWGN
- Optimum Detection with Nusiance (Unwanted) Parameters

M-ary Detection Problem

Consider the problem of deciding which of M hypothesis is true based on observing a random variable (vector) r. The performance criteria we consider is the average error probability. That is the probability of deciding anything except hypothesis H_j when hypothesis H_j is true.

The underlying model is that there is a conditional probability density (mass) function of the observation r given each hypothesis H_j.

$$\begin{aligned} E[P_e] &= \sum_{i=0}^{M-1}\left[\sum_{j\neq i} P\{\text{decide } H_j | H_i \text{ true}\}\right]\pi_i \\ &= \sum_{i=0}^{M-1}[1-P\{\text{decide } H_i | H_i \text{ true}\}]\pi_i \\ &= \sum_{i=0}^{M-1}\pi_i - \sum_{i=0}^{M-1}\int_{R_i} p_i(x)\pi_i dx \\ &= 1-\sum_{i=0}^{M-1}\int_{R_i} p_i(x)\pi_i dx \ . \end{aligned}$$

The decision rule that minimizes average cost assigns x to R_i if $p_i(x)\pi_i = \max_{0\le j\le M-1} p_j(x)\pi_j$.

Let $p(x)$ be an arbitrary density function that is nonzero everywhere $p_i(x)$ is nonzero then an equivalent decision rule is to assign x to R_i if

$$\frac{p_i(x)}{p(x)}\pi_i = \max_{0\le j\le M-1}\frac{p_j(x)}{p(x)}\pi_j.$$

Thus for M hypotheses the decision rule that minimizes average error probability is to choose i so that $p_i(x)\pi_i > p_j(x)\pi_j, \ \forall \ j\neq i$. Let

$$\Lambda_{i,j} = \frac{p_i(x)}{p_j(x)}$$

where $i = 0,1,\ldots,M-1$, $j = 0,1,\ldots,M-1$. Then the optimal decision rule is:

$$\text{Choose } i \text{ if } \Lambda_{i,j} > \frac{\pi_j}{\pi_i} \text{ for all } j\neq i.$$

We will usually assume $\pi_i = \frac{1}{M}\ \forall i$. (If not we should do source encoding to reduce the entropy (rate)). For this case the optimal decision rule is

$$\text{Choose } i \text{ if} \qquad \Lambda_{i,j} > 1 \ \ \forall \ \ j\neq i.$$

Example 1: Additve White Gaussian Noise

Consider three signals in additive white Gaussian noise. For additive white Gaussian noise $K(s,t) = \frac{N_0}{2}\delta(t-s)$. Let $\{\varphi_i(t)\}_{i=0}^{\infty}$ be any complete orthonormal set on $[0,T]$. Consider the case of 3 signals. Find the decision rule to minimize average error probability. First expand the noise using orthonormal set of functions and random variables.

$$n(t) = \sum_{i=0}^{\infty} n_i\varphi_i(t)$$

where $E[n_i] = 0$ and $\text{Var}[n_i] = N_0/2$ and $\{n_i\}_{i=0}^{\infty}$ is an independent identically distributed (i.i.d.) sequence of random variables with Gaussian density functions.

Let

$$\begin{aligned} s_0(t) &= \varphi_0(t)+2\varphi_1(t) \\ s_1(t) &= 2\varphi_0(t)+\varphi_1(t) \\ s_2(t) &= \varphi_0(t)-2\varphi_1(t) \end{aligned}$$

Note that the energy of each of the three signals is the same, i.e. $\int_0^T s_i^2(t)dt = ||s_i||^2 = 5$. Then

we have a three hypothesis testing problem.

$$\begin{aligned} H_0 : r(t) &= s_0(t) + n(t) = \sum_{i=0}^{\infty} (s_{0,i} + n_i)\varphi_i(t) \\ H_1 : r(t) &= s_1(t) + n(t) = \sum_{i=0}^{\infty} (s_{1,i} + n_i)\varphi_i(t) \\ H_2 : r(t) &= s_2(t) + n(t) = \sum_{i=0}^{\infty} (s_{2,i} + n_i)\varphi_i(t) \end{aligned}$$

The decision rule to minimize the average error probability is given as follows

$$\text{Decide } H_i \text{ if } \pi_i p_i(\mathbf{r}) = \max_j \pi_j p_j(\mathbf{r})$$

First let us normalize each side by the density function for the noise alone. The noise density function for $N+1$ variables is

$$p^{(N)}(\mathbf{r}) = \left(\frac{1}{\sqrt{2\pi N_0/2}}\right)^N \exp\{-\frac{1}{2\frac{N_0}{2}} \sum_{i=0}^{N} r_i^2\}$$

The the optimal decision rule is equivalent to

$$\text{Decide } H_i \text{ if } \pi_i \frac{p_i(\mathbf{r})}{p(\mathbf{r})} = \max_j \pi_j \frac{p_j(\mathbf{r})}{p(\mathbf{r})}.$$

As usual assume $\pi_i = 1/M$. Then

$$\begin{aligned} \frac{p_0^{(N)}(\mathbf{r})}{p^{(N)}(\mathbf{r})} &= \frac{\left(\frac{1}{\sqrt{2\pi N_0/2}}\right)^N \exp\{-\frac{1}{2\frac{N_0}{2}}[\sum_{i=0,1}(r_i - s_{0,i})^2 + \sum_{i=2}^{N} r_i^2]\}}{\left(\frac{1}{\sqrt{2\pi N_0/2}}\right)^N \exp\{-\frac{1}{2\frac{N_0}{2}} \sum_{i=0}^{1} r_i^2 + \sum_{i=2}^{N}\}} \\ &= \exp\{-\frac{1}{N_0}[\sum_{i=0,1}(r_i - s_{0,i})^2 - r_i^2]\} \\ &= \exp\{+\frac{1}{N_0}[2r_1 + 4r_2 - 5]\}. \end{aligned}$$

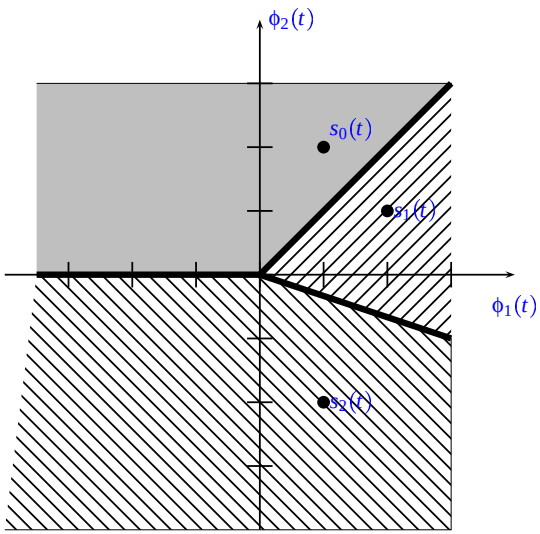
Now since the above doesn't depend on N we can let $N \to \infty$ and the result is the same, i.e.

$$\frac{p_0(\mathbf{r})}{p(\mathbf{r})} \triangleq \lim_{N\to\infty} \frac{p_0^{(N)}(\mathbf{r})}{p^{(N)}(\mathbf{r})} = \exp\{+\frac{1}{N_0}[2r_1 + 4r_2 - 5]\}.$$

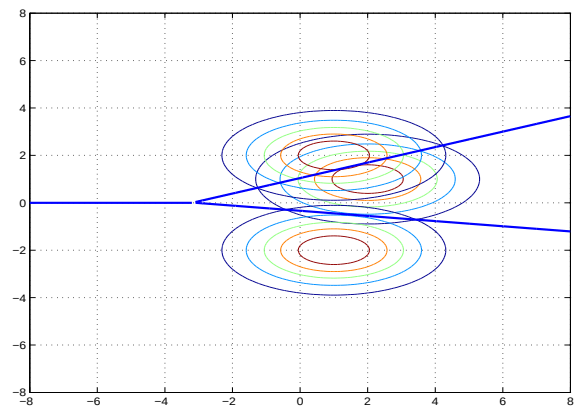
Similarly

$$\frac{p_1(\mathbf{r})}{p(\mathbf{r})} = \exp\{+\frac{1}{N_0}[4r_1 + 2r_2 - 5]\}$$

$$\frac{p_2(\mathbf{r})}{p(\mathbf{r})} = \exp\{+\frac{1}{N_0}[2r_1 - 4r_2 - 5]\}.$$



Decision Regions



Example 2: Optimum Detection of M-ary orthogonal signals for minimum bit error probability

In this section we consider the problem of detection with unwanted parameters. To illustrate consider the problem of minimizing the bit error probability in an M-ary orthogonal signal set.

Let $s_0(t), ..., s_{M-1}(t)$ be orthogonal signals.

$$\begin{array}{lll} 00000 & s_0(t) & = \sqrt{E}\phi_0(t) \\ 00001 & s_1(t) & = \sqrt{E}\phi_1(t) \\ . & . & \\ . & . & \\ . & . & \\ 11111 & s_{M-1}(t) & = \sqrt{E}\phi_{M-1}(t) \end{array}$$

Let $b_0, ..., b_{k-1}$ be the sequence of bits determining which of the M signals is transmitted. Assume the bits are independent and equally likely.

The receiver consists of a bank of matched filters (correlators) that generate a sufficient statistic. If signal s_j is transmitted then

$$\begin{array}{rcl} r_0 & = & \delta(j,0)\sqrt{E} + \eta_0 \\ r_1 & = & \delta(j,1)\sqrt{E} + \eta_1 \\ . & . & \\ . & . & \\ . & . & \\ . & . & \\ r_{M-1} & = & \delta(j,M-1)\sqrt{E} + \eta_{M-1} \end{array}$$

Consider the detection of data bit b_0. That is, we are interested in minimizing the probability of error for data bit b_0. Let H_0 be the event that $b_0 = 0$ and H_1 be the event that $b_0 = 1$. Let $r = (r_0, r_1, ...r_{M-1})$. Then the optimal receiver must compare the two aposteriori probabilities

$$p(r|H_0)\pi_0 \underset{H_1}{\overset{H_0}{\gtrless}} p(r|H_1)\pi_1$$

To calculate $p(r|H_0)$ we proceed as follows.

$$\begin{aligned} p(r|H_0)\pi_0 &= p(r|b_0 = 0)\pi_0 \\ &= \pi_0 \sum_{b_1,...,b_{k-1}} p(r|b_0 = 0, b_1, ...b_{k-1}) p(b_1)p(b_2)\cdots p(b_{k-1}) \\ &= 2^{-k} \sum_{b_1,...,b_{k-1}} (\frac{1}{\sqrt{2\pi}\sigma})^M \exp\{-\frac{1}{2\sigma^2}\sum_{l=0}^{M-1}(r_l - s(b))^2\} \\ &= 2^{-k} \sum_{m=0}^{M/2-1} (\frac{1}{\sqrt{2\pi}\sigma})^M \exp\{-\frac{1}{2\sigma^2}\sum_{l=0}^{M-1}(r_l - \delta(l,m)\sqrt{E})^2\} \\ &= 2^{-k}(\frac{1}{\sqrt{2\pi}\sigma})^M \sum_{m=0}^{M/2-1} \exp\{-\frac{1}{2\sigma^2}\sum_{l=0}^{M-1}(r_l - \delta(l,m)\sqrt{E})^2\} \\ &= 2^{-k}(\frac{1}{\sqrt{2\pi}\sigma})^M \sum_{m=0}^{M/2-1} \exp\{-\frac{1}{2\sigma^2}\sum_{l=0}^{M-1}(r_l^2 - 2r_l\delta(l,m)\sqrt{E} + \delta(l,m)E\} \\ &= 2^{-k}(\frac{1}{\sqrt{2\pi}\sigma})^M \exp\{-\frac{1}{2\sigma^2}\sum_{l=0}^{M-1} r_l^2\} \exp\{-E/2\sigma^2\} \sum_{m=0}^{M/2-1} \exp\{\frac{1}{\sigma^2}\sum_{l=0}^{M-1}(r_l\delta(l,m)\sqrt{E}\} \\ &= 2^{-k}(\frac{1}{\sqrt{2\pi}\sigma})^M \exp\{-\frac{1}{2\sigma^2}\sum_{l=0}^{M-1} r_l^2\} \exp\{-E/2\sigma^2\} \sum_{m=0}^{M/2-1} \exp\{\frac{r_m\sqrt{E}}{\sigma^2}\} \end{aligned}$$

Similarly

$$\begin{aligned} p(r|H_1)\pi_1 &= p(r|b_0 = 1)\pi_1 \\ &= 2^{-k}(\frac{1}{\sqrt{2\pi}\sigma})^M \exp\{-\frac{1}{2\sigma^2}\sum_{l=0}^{M-1} r_l^2\} \exp\{-E/2\sigma^2\} \sum_{m=M/2}^{M-1} \exp\{\frac{r_m\sqrt{E}}{\sigma^2}\} \end{aligned}$$

Notice that many of the factors in $p(r|H_1)\pi_1$ and $p(r|H_1)\pi_1$ are the same. Thus the likelihood ratio for bit b_0 is

$$\frac{p(r|H_1)\pi_1}{p(r|H_0)\pi_0} = \frac{\sum_{m=M/2}^{M-1} \exp\{\frac{r_m\sqrt{E}}{\sigma^2}\}}{\sum_{m=0}^{M/2-1} \exp\{\frac{r_m\sqrt{E}}{\sigma^2}\}}$$

The log-likelihood ratio is

$$\log(\frac{p(r|H_1)\pi_1}{p(r|H_0)\pi_0}) = \log(\sum_{m=M/2}^{M-1} \exp\{\frac{r_m\sqrt{E}}{\sigma^2}\}) - \log(\sum_{m=0}^{M/2-1} \exp\{\frac{r_m\sqrt{E}}{\sigma^2}\})$$

This can be approximated by

$$\log(\frac{p(r|H_1)\pi_1}{p(r|H_0)\pi_0}) \approx \max_{m=M/2}^{M-1}(r_m\sqrt{E}/\sigma^2) - \max_{m=0}^{M/2-1}(r_m\sqrt{E}/\sigma^2)$$

Example 3: Optimum Detection of binary signals in fading channels

Consider a system with L antennas. Assume that the receiver knows exactly the faded amplitude on each antenna. The decision statistics are then given by

$$z_l = r_l\sqrt{E}b + \eta_l, \quad l = 1,2,...,L$$

where r_l are Rayleigh, η_l is Gaussian and b represents the data bit transmitted which is either +1 or -1.

The random variable r_l represents the fading from the transmitter to the $l-th$ antenna and has density

$$p(r_l) = \begin{cases} 0, & r < 0 \\ \frac{r}{\sigma^2}e^{-r^2/2\sigma^2} & r \geq 0 \end{cases}$$

We assume the fading to each antenna is independent. The optimal method to combine the demodulator outputs can be derived as follows. Let $p_1(z_1,...,z_L|r_1,...,r_L)$ be the conditional density function of $z_1,...,z_L$ given the transmitted bit is +1 and the fading amplitude is

$r_1,...,r_L$. The unconditional density is

$$p_1(z_1,...,z_L,r_1,...,r_L) = p_1(z_1,...,z_L|r_1,...,r_L)p(r_1,...,r_L)$$

The conditional density of z_1 given $b = 1$ and r_1, is Gaussian with mean $r_l\sqrt{E}$ and variance $N_0/2$. The joint distribution of $z_1,...,z_L$ is the product of the marginal density functions. The optimal combining rule is derived from the ratio

$$\begin{aligned} \Lambda &= \frac{p_1(z_1,...,z_L,r_1,...,r_L)}{p_{-1}(z_1,...,z_L,r_1,...,r_L)} \\ &= \frac{p_1(z_1,...,z_L|r_1,...,r_L)p(r_1,...,r_L)}{p_{-1}(z_1,...,z_L|r_1,...,r_L)p(r_1,...,r_L)} \\ &= \frac{p_1(z_1,...,z_L|r_1,...,r_L)}{p_{-1}(z_1,...,z_L|r_1,...,r_L)} \\ &= \frac{\exp\{-\frac{1}{N_0}\sum_{l=1}^{L}(z_l - r_l\sqrt{E})^2)}{\exp\{-\frac{1}{N_0}\sum_{l=1}^{L}(z_l + r_l\sqrt{E})^2)} \\ &= \exp\{\frac{4}{N_0}\sum_{l=1}^{L} z_l r_l\sqrt{E}\}. \end{aligned}$$

The optimum decision rule is to compare Λ with 1 to make a decision. Thus the optimal rule is

$$\sum_{l=1}^{L} r_l z_l \underset{b=-1}{\overset{b=+1}{\gtrless}} 0$$

Note that we do not need to know the denisty of the amplitude for this decision rule. This decision rule is called maximum ratio combining (MRC). In the special case where there is just one antenna the optimum receiver reduces to

$$r_1 z_1 \underset{b=-1}{\overset{b=+1}{\gtrless}} 0 \quad \Rightarrow \quad z_1 \underset{b=-1}{\overset{b=+1}{\gtrless}} 0$$

Thus the optimum receiver for just one antenna (and BPSK) does not need the informaiton about the received amplitude to make a (hard) decision. However, the performance depends critically on the distribution of the fading amplitude. For the Rayleigh faded case the error probability becomes

$$P_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\bar{E}/N_0}{1+E/N_0}}$$

Likelihood Ratio for Real Signals in AGN

Assume two signals in Gaussian noise.

$$\begin{aligned} H_0 &: \quad r(t) = s_0(t) + n(t) \\ H_1 &: \quad r(t) = s_1(t) + n(t) \end{aligned}$$

Goal: Find decision rule to minimize the average error probability.

Let $n(t)$ have covariance $K(s,t)$ with eigenfunction $\varphi_i(t)$ and eigenvalues λ_i. We assume that $n(t)$ is a zero mean Gaussian random process. The eigen functions φ_i are orthonormal functions and λ_i real numbers such that (see Appendix)

$$\int K(s,t)\varphi_i(t)dt = \lambda_i\varphi_i(s)$$

Karhunen-Loeve Expansion

By Karhunen-Loeve expansion

$$n(t) = \sum_{i=1}^{\infty} n_i \varphi_i(t)$$

where n_i are Gaussian random variables with mean 0 variance λ_i and $E[n_i n_j] = 0 \Rightarrow n_i, n_j$ independent ($n(t)$ is real). Since $\varphi_i(t)$ are a complete orthonormal set and we assume $s_j(t)$ has finite energy we have $s_j(t) = \sum_{i=0}^{\infty} s_{j,i}\varphi_i(t)$.

Thus

$$\begin{aligned} H_j : r(t) &= \sum_{i=0}^{\infty} (s_{j,i} + n_i)\varphi_i(t) \\ & \quad r_i = s_{j,i} + n_i, \quad i = 1, 2, \ldots \end{aligned}$$

Define

$$\Lambda_{j,i}(N) \triangleq \frac{p_j(r_1, r_2, \ldots, r_N)}{p_i(r_1, r_2, \ldots, r_N)} \quad .$$

$$\Lambda_{j,i}(r(t)) \triangleq \lim_{N\to\infty} \Lambda_{ji}(N)$$

where r_i is Gaussian mean $s_{j,i}$ variance λ_i.

$$\begin{aligned} p_j(r_i) &= \frac{1}{\sqrt{2\pi}\lambda_i} \exp\left\{-\frac{1}{2\lambda_i}(r_i - s_{j,i})^2\right\} \\ p_j(\underline{r}) = \prod_{i=0}^{N} p_j(r_i) &= \prod_{i=0}^{n} (\sqrt{2\pi}\lambda_i)^{-1} \exp\left\{-\frac{1}{2}\sum_{i=0}^{N} \frac{(r_i - s_{j,i})^2}{\lambda_i}\right\} \\ \Lambda_{j,l}(N) = \frac{p_j^N(\underline{r})}{p_l^N(\underline{r})} &= \frac{\prod_{i=0}^{N} (\sqrt{2\pi\lambda_i})^{-1} \ \exp\left\{-\frac{1}{2}\sum_{i=0}^{N} \frac{(r_i - s_{j,i})^2}{\lambda_i}\right\}}{\prod_{i=0}^{N} (\sqrt{2\pi}\lambda_i)^{-1} \ \exp\left\{-\frac{1}{2}\sum_{i=0}^{N} \frac{(r_i - s_{l,i})^2}{\lambda_i}\right\}} \\ &= \exp\left\{-\frac{1}{2}\sum_{i=0}^{N} \frac{1}{\lambda_i}[r_i^2 - 2r_i s_{j,i} + s_{j,i}^2 - r_i^2 + 2r_i s_{l,i} - s_{l,i}^2]\right\} \\ &= \exp\left\{-\frac{1}{2}\sum_{i=0}^{N} \frac{1}{\lambda_i}[s_{j,i}^2 - s_{l,i}^2 + 2r_i(s_{l,i} - s_{j,i})]\right\}. \end{aligned}$$

Let

$$q_j(t) = \lim_{N\to\infty} \sum_{i=0}^{N} \frac{s_{j,i}}{\lambda_i}\varphi_i(t) = \sum_{i=0}^{\infty} \frac{s_{j,i}\varphi_i(t)}{\lambda_i}$$

Then

$$\begin{aligned} \int r(t) q_j(t) dt &= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty} \int r_l \varphi_l(t) \frac{s_{j,i}}{\lambda_i}\varphi_i(t) dt \\ &= \sum_{i=0}^{\infty} \frac{r_i s_{j,i}}{\lambda_i} \end{aligned}$$

$$\begin{aligned} \int s_j(t) q_j(t) dt &= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty} \int s_{j,i}\varphi_i(t) \frac{s_{j,l}}{\lambda_l}\varphi_l(t) dt \\ &= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty} \frac{s_{j,i}s_{j,l}}{\lambda_l} \int \varphi_i(t)\varphi_l(t) dt \\ &= \sum_{i=0}^{\infty} \frac{s_{j,i}^2}{\lambda_i} = (s_j, q_j). \end{aligned}$$

Thus

$$\Lambda_{j,l}(r(t)) = \lim_{N\to\infty} \Lambda_{j,l}(N) = \exp\left\{-\frac{1}{2}[(s_j, q_j) - (s_l, q_l) + 2(r, q_l) - 2(r, q_j)]\right\}.$$

Note: $q_j(t)$ is solution of the integral equation

$$\int K(s,t) q_j(t) dt = s_j(s)$$

$$\Downarrow$$

$$q_j(t) = \sum_{i=0}^{\infty} \frac{s_{j,i}}{\lambda_i} \varphi_i(t).$$

So

$$q_j(s) = \int K^{-1}(s,t) s_j(t) dt$$

$$q_j = K^{-1} s_j$$

If the noise is white, then the noise power in each direction is constant (say λ) and thus

$$q_j(t) = \sum_{i=0}^{\infty} \frac{s_{j,i}}{\lambda} \varphi_i(t) = \frac{1}{\lambda} s_j(t).$$

The optimal receiver then becomes

$$\Lambda_{j,l}(r(t)) = \exp\left\{-\frac{1}{2\lambda}[(s_j, s_j) - (s_l, s_l) + 2(r, s_l) - 2(r, s_j)]\right\}.$$

or equivalently

$$\Lambda_{j,l}(r(t)) = \exp\left\{-\frac{1}{2\lambda}[||s_j||^2 - ||s_l||^2 + 2(r, s_l - s_j)]\right\}.$$

For equal energy signals this amounts to picking the signal with the largest correlation with the received signal.

The optimal receiver in nonwhite Gaussian noise can be implemented in a similar fashion as shown below.

$$\begin{aligned}(s_j,q_j) &= (s_j,K^{-1}s_j) = (K^{-1/2}s_j,K^{-1/2}s_j) = \|K^{1/2}s_j\|^2\\ (r,q_j) &= (r,K^{-1}s_j) = (K^{-1/2}r,K^{-1/2}s_j)\end{aligned}$$

Thus

$$\Lambda_{j,l}(r(t)) = \exp\left\{-\frac{1}{2}[\|K^{-1/2}s_j\|^2 - \|K^{-1/2}s_l\|^2 + 2(K^{-1/2}r,K^{-1/2}(s_l-s_j))]\right\}.$$

It is clear then that this is just the optimal filter for signals $K^{-1/2}s_j$ when received in additive white Gaussian noise. This approach is called "whitening" because $K^{-1/2}n$ will be a white Gaussian noise process.

Likelihood Ratio for Complex Signals

In this section we rederive the likelihood ratios for complex signals received in complex noise. We assume that the signals are the lowpass representation of bandpass signal and the noise is the lowpass representation of a narrowband random process. Let

$$\begin{aligned}H_0 : r(t) &= s_0(t)+n(t)\\ H_1 : r(t) &= s_1(t)+n(t)\end{aligned}$$

where $n(t)$ has covariance $K(s,t)$, with eigenfunctions $\varphi_i(t)$, eigenvalues λ_i. Using Karhunen-Loeve expansion we have

$$H_i : r(t) = \sum_{j=0}^{\infty}(s_{i,j}+n_j)\varphi_j(t)$$

$$r_j = s_{i,j}+n_j.$$

$$\begin{aligned}\frac{p_j(r_1,\ldots,r_n)}{p_i(r_1,\ldots,r_n)} &= \frac{\prod_{l=0}^{n}\frac{1}{\pi\lambda_l}\,e^{-(|r_l-s_{j,l}|^2)/\lambda_l}}{\prod_{l=0}^{n}\frac{1}{\pi\lambda_l}\,e^{-(|r_l-s_{i,l}|^2)/\lambda_l}}\\ &= \prod_{l=0}^{n}\exp\{-(|r_l-s_{j,l}|^2-|r_l-s_{il}|^2)/\lambda_l\}\end{aligned}$$

$$\begin{aligned}&= \exp\left\{-\sum_{l=0}^{n}(|r_l-s_{j,l}|^2-|r_l-s_{i,l}|^2)/\lambda_l\right\}\\ &= \exp\left\{-\sum_{l=0}^{n}\frac{|r_l|^2+|s_{j,l}|^2-2\,\mathrm{Re}\,(r_ls^*_{j,l})-|r_l|^2-|s_{i,l}|^2+2\,\mathrm{Re}\,(r_ls^*_{i,l})}{\lambda_l}\right\}\\ &= \exp\left\{-\sum_{l=0}^{n}\frac{|s_{j,l}|^2-|s_{i,l}|^2+2\mathrm{Re}\,[r_l(s_{i,l}-s_{j,l})^*]}{\lambda_l}\right\}.\end{aligned}$$

Let $q_j(t) = \sum_{i=0}^{\infty}\frac{s_{ji}}{\lambda_i}\varphi_i(t)$ then

$$\begin{aligned}(q_j(t),s_j(t)) &= \int_a^b\sum_{l=0}^{\infty}\frac{s_{j,l}}{\lambda_l}\varphi_l(t)\sum_{k=0}^{\infty}s^*_{j,k}\varphi^*_k(t)dt\\ &= \sum_{l=0}^{\infty}|s_{j,l}|^2/\lambda_l = (s_j(t),q_j(t))\end{aligned}$$

$$\begin{aligned}(r(t),q_j(t)) &= \int_a^b\sum_{l=0}^{\infty}r_l\varphi_l(t)\sum_{k=0}^{\infty}\frac{s^*_{j,k}}{\lambda_k}\,\varphi^*_k(t)dt\\ &= \sum_{l=0}^{\infty}\frac{r_ls^*_{j,l}}{\lambda_l}.\end{aligned}$$

So

$$\begin{aligned}\Lambda_{ji}(r(t)) &= \lim_{H\to\infty}\Lambda^{(n)}_{ji}(r(t)) = \frac{p_j(r_1,\ldots,r_n)}{p_i(r_1,\ldots,r_n)}\\ &= \exp\{-[(s_j,q_j)-(s_i,q_i)+2\,\mathrm{Re}\,(r(t),q_i(t)-q_j(t))]\}\end{aligned}$$

Note: Since we are dealing with noise that is derived from a narrowband random process we can not use the results derived for *real* random processes we *must* use the likelihood ratio for *complex* random process given above.

For real random process the likelihood ratio is

$$\Lambda_{ji}(N) = \exp\{-\frac{1}{2}[(s_j,q_j)-(s_i,q_i)+2(r,q_i-q_j)]\}.$$

For additive white Gaussian noise (real)

$$\begin{aligned}q_i(t) = \sum_{j=1}^{\infty}\frac{s_{i,j}\varphi_j(t)}{\lambda_j} &= \frac{2}{N_0}\sum_{j=1}^{\infty}s_{i,j}\varphi_j(t)\\ &= \frac{2}{N_0}s_i(t)\end{aligned}$$

So the likelihood ratio (for real signals) becomes

$$\begin{aligned}
\Lambda_{j,l} &= \lim_{N\to\infty} \frac{p_j(r_1,\ldots,r_N)}{p_l(r_1,\ldots,r_N)} = \exp\left\{-\frac{1}{2}\left[\frac{2}{N_0}((s_j,s_j)-(s_l,s_l))+2\cdot\frac{2}{N_0}(r,s_l-s_j)\right]\right\} \\
&= \exp\left\{-\frac{1}{N_0}[\|s_j\|^2-2(r,s_j)-\|r\|^2-(\|s_l\|^2-2(r,s_l)-\|r\|^2)]\right\} \\
&= \exp\left\{-\frac{1}{N_0}[\|s_j-r\|^2-\|s_l-r\|^2]\right\} \underset{H_l}{\overset{H_j}{\gtrless}} 1.
\end{aligned}$$

Assume $\pi_j = \frac{1}{M} \quad j = 1,2,\ldots,M$. Then $\alpha = 1$. An equivalent decision rule then is

$$\|s_j-r\|^2-\|s_l-r\|^2 \underset{H_j}{\overset{H_l}{\gtrless}} 0$$

$$\|s_j-r\|^2 \underset{H_j}{\overset{H_l}{\gtrless}} \|s_l-r\|^2.$$

The optimum decision rule for additive white Gaussian noise is then to choose i if

$$\|s_i-r\|^2 = \min_{1\le j\le M} \|s_j-r\|^2.$$

Example: M orthogonal signals in additive white Gaussian noise

In this section we consider the optimum receiver for M-ary orthogonal signals and the associated error probaiblity. Assume the M signals are equienergy signals and equiprobable. The decision rule derived previously for AWGN is

$$\text{Decide } H_i \text{ if } \|s_i-r\|^2 = \min_{1\le j\le M} \|s_j-r\|^2.$$

Now since the M signals are orthogonal and equienergy we can write this as

$$\|s_j-r\|^2 = \|s_j\|^2-2(s_j,r)+\|r\|^2.$$

The first term above is constant for each j as is the last term. Thus finding the minimum is equivalent to finding the maximum of

$$(s_j,r).$$

Thus the receiver should compute the inner product between the M different signals and find the largest such correlation. If the signals are all of duration T, i.e. zero outside the interval $[0,T]$ then this is also equivalent to filtering the received signal with a filter with impluse response $s_j(T-t)$, sampling the output of the filter at time T and choosing the largest as shown below.