Lecture Notes 3: Error Probability for M Signals

Goals

1. Exact analysis of M-ary orthogonal signals in AWGN channels.
2. Gallager bound for arbitrary signals, arbitrary channel.
3. Random Coding Bound.

Error Probability

Problem: Determine the error probability in deciding which of M signals was transmitted over an arbitrary channel with some transition probability

$$p(r_0,...,r_{N-1}|\mathbf{s}_i \text{ transmitted}), \; i = 0,1,...,M-1.$$

Writing down an expression for the error probability in terms of an N-dimensional integral is straightforward. However, evaluating the integrals involved in the expression in all but a few special cases is very difficult or impossible if N is fairly large (e.g. $N > 4$).

Summary of Bounds

For the special case of orthogonal signals the error probability can be expressed as a single integral. Because of the difficulty of evaluating the error probability for general signal sets, bounds are needed to determine the performance. Different bounds have different complexity of evaluation. This first bound we derive is known as the Gallager bound. We apply this bound to the case of orthogonal signals (for which the true answer is already known). The Gallager bound has the property that when the number of signals become large the bound becomes tight. However, the bound is fairly difficult to evaluate for many signal sets. A special case of the Gallager bound is the Union-Bhattacharayya bound. This is simpler than the Gallager bound to evaluate but also is looser than the Gallager bound. The last bound considered is the union bound. This bound is tighter than the Union-Bhattacharayya bound and the Gallager bound for sufficiently high signal-to-noise ratios.

Random Coding Bound

Finally we consider a simple random coding bound on the ensemble of all signal sets using the Union-Bhattacharayya bound.

Error Probability for M Signals

The general signal set we consider has the form

$$s_i(t) = \sum_{j=0}^{N-1} s_{i,j}\varphi_j(t), \quad i = 0, 1, \ldots, M-1$$

The number of othonormal basis functions is less than the number of signals ($N < M$). The optimum receiver does a correlation with the N orthonormal waveforms to form the decision variables.

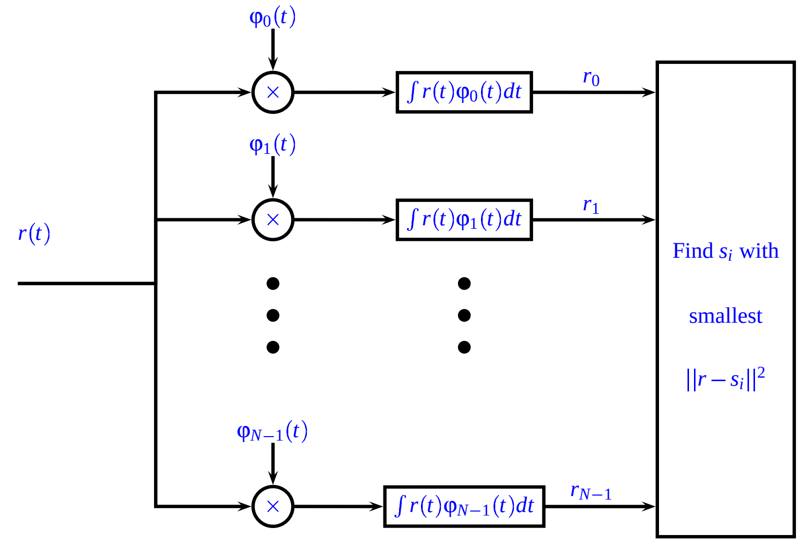


Figure 9: Optimum Receiver in Additive White Gaussian Noise

$$r_j = \int_0^T r(t)\varphi_j(t)dt, j = 0, 1, \ldots, N-1.$$

The decision regions are for equaly likely signals given by

$$R_i \triangleq \{\mathbf{r} : p_i(\mathbf{r}) > p_j(\mathbf{r}), \forall j \neq i\}$$

The error probability is then determined by

$$P_{e,i} = P(\bigcup_{j=0, j\neq i}^{M-1} R_j | H_i).$$

For all but a few small dimensional signals or signals with special structures (such as orthogonal signal sets) the exact error probability is very difficult to calculate.

Error Probability for Orthogonal Signals

Represent the M signals in terms of M orthogonal function $\varphi_i(t)$ as follows

$$\begin{aligned} s_0(t) &= \sqrt{E}\varphi_0(t) \\ s_1(t) &= \sqrt{E}\varphi_1(t) \\ . &\quad . \\ . &\quad . \\ . &\quad . \\ s_{M-1}(t) &= \sqrt{E}\varphi_{M-1}(t). \end{aligned}$$

The optimal receiver finds the largest value of $(r(t), s_j(t))$ for $j = 0, \ldots, M-1$. Equivalently the optimum receiver determines the largest value of $r_j = (r(t), s_j(t))/\sqrt{E}$.

Statistics of r_j

To determine the error probability we need to determine the statistics of r_j. Assume signal s_i is transmitted. Then

$$r_j \triangleq \int_0^T r(t)\varphi_j(t)dt$$

$$\begin{aligned} E[r_j|H_i] &= E[\int_0^T r(t)\varphi_j(t)dt|H_i] \\ &= \int_0^T E[r(t)|H_i]\varphi_j(t)dt \\ &= \int_0^T E[s_i(t)+n(t)]\varphi_j(t)dt \\ &= \int_0^T \sqrt{E}\varphi_i(t)\varphi_j(t)dt \\ &= \sqrt{E}\delta_{i,j}. \end{aligned}$$

Variance of r_j

The variance of r_j is determined as follows. Given H_i

$$\begin{aligned} r_j - E[r_j|H_i] &= \int_0^T n(t)\varphi_j(t)dt \\ E[(r_j - E[r_j|H_i])^2|H_i] &= \int_0^T \int_0^T E[n(t)n(s)]\varphi_j(t)\varphi_j(s)dtds \\ &= \int_0^T \int_0^T K(t,s)\varphi_j(t)\varphi_j(s)dtds \\ &= \int_0^T \int_0^T \frac{N_0}{2}\delta(t-s)\varphi_j(t)\varphi_j(s)dtds \\ &= \int_0^T \frac{N_0}{2}\varphi_j(t)\varphi_j(t)dt \\ &= \frac{N_0}{2}. \end{aligned}$$

Furthermore, each of these random variables is Gaussian (and independent).

Error Probability

$$P(\text{error}) = 1 - P(\text{correct})$$

$$P(\text{correct}) = \sum_{i=0}^{M-1} P(H_i|H_i)\pi_i$$

$$\begin{aligned} P_{c,i} \triangleq P(H_i|H_i) &= P(r_i > r_j,\ \forall j \neq i|H_i) \\ &= E[P(r_i > r_j,\ \forall j \neq i|H_i, r_i)] \\ &= E[\prod_{j=0, j\neq i}^{M-1} P(r_i > r_j|H_i, r_i)] \\ &= E[\prod_{j=0, j\neq i}^{M-1} \Phi(\frac{r_i}{\sqrt{N_0/2}})] \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp\{-\frac{1}{N_0}(r_i - \sqrt{E})^2\}\Phi^{M-1}(\frac{r_i}{\sqrt{N_0/2}})dr_i. \end{aligned}$$

$\left(\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du\right)$. Now let $u = \frac{r_i}{\sqrt{N_0/2}}$. Then

$$\begin{aligned} P_{e,i} = 1 - P_{c,i} &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi N_0}} \sqrt{\frac{N_0}{2}} \exp\{-(\sqrt{\frac{N_0}{2}}u - \sqrt{E})^2/N_0\}[1 - \Phi^{M-1}(u)]du \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-(u - \sqrt{\frac{2E}{N_0}})^2/2\}[1 - \Phi^{M-1}(u)]du \\ &= (M-1)\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \Phi(u - \sqrt{\frac{2E}{N_0}})\Phi^{M-2}(u)e^{-u^2/2}du \end{aligned}$$

where the last step follows from using an integration by parts approach. Later on we will find an upper bound on the above that is more insightful. It is possible to determine (using L'Hospital's rule) the limiting behavior of the error probability as $M \to \infty$.

In general if we have M decision variables for an M-ary hypothesis testing problem that are conditionally independent given the true hypothesis and there is a density (distribution) of the decision variable for the true statistic denoted $f_1(x)$ ($F_1(x)$) and a density and distribution function for the other decision variables ($f_2(x), F_2(x)$) then the probability of correct is

$$P_c = \int_{-\infty}^{\infty} f_1(x) F_2^{M-1}(x)dx$$

The probability of error is

$$P_e = 1 - \int_{-\infty}^{\infty} f_1(x) F_2^{M-1}(x)dx$$
$$= (M-1)\int_{-\infty}^{\infty} F_1(x) F_2^{M-2}(x) f_2(x)dx$$

The last formula is many times easier to compute numerically than the first because the former is the difference between two numbers that are very close (for small error probabilities).

Gallager Bound

Now we derive an upper bound on the error probability for M signals received in some form of noise. Let $\mathbf{r}$ be an N-dimensional noise vector.

$$R_i \triangleq \{\mathbf{r} : p_i(\mathbf{r}) > p_j(\mathbf{r}), \forall j \neq i\}$$
$$\overline{R}_i \triangleq \{\mathbf{r} : p_i(\mathbf{r}) \leq p_j(\mathbf{r}), \text{for some } j \neq i\}$$
$$P_{c,i} \triangleq P(H_i|H_i)$$
$$P_{e,i} \triangleq P(\overline{H}_i|H_i) = P(\overline{R}_i|H_i).$$

Now

$$\overline{R}_i \triangleq \{\mathbf{r} : \frac{p_j(\mathbf{r})}{p_i(\mathbf{r})} \geq 1, \text{ for some } j \neq i\}.$$

For $\lambda \geq 0$ let

$$\widetilde{R}_i \triangleq \{\mathbf{r} : \sum_{j \neq i} \left[\frac{p_j(\mathbf{r})}{p_i(\mathbf{r})}\right]^{\lambda} \geq 1\}.$$

Gallager Bound (cont.)

Claim: Then $\widetilde{R}_i \supset \overline{R}_i$. **Proof:** If $\mathbf{r} \in \overline{R}_i$ then $\frac{p_j(\mathbf{r})}{p_i(\mathbf{r})} \geq 1$ for some $j \neq i$. Thus for some $j \neq i$, $\left[\frac{p_j(\mathbf{r})}{p_i(\mathbf{r})}\right]^{\lambda} \geq 1$ which implies that

$$\sum_{j \neq i} \left[\frac{p_j(\mathbf{r})}{p_i(\mathbf{r})}\right]^{\lambda} \geq 1$$

and thus $\mathbf{r} \in \widetilde{R}_i$. Thus we have shown that $\widetilde{R}_i \supset \overline{R}_i$. Now we use this to upper bound the error probability.

$$P_{e,i} = P(\overline{R}_i|H_i) \leq P(\widetilde{R}_i|H_i) = \int_{\widetilde{R}_i} p_i(\mathbf{r})d\mathbf{r}$$
$$= \int_{R^M} I[\widetilde{R}_i] p_i(\mathbf{r})d\mathbf{r}$$

where

$$I[\widetilde{R}_i] = \begin{cases} 1, & \mathbf{r} \in \widetilde{R}_i \\ 0, & \mathbf{r} \notin \widetilde{R}_i. \end{cases}$$

For $\mathbf{r} \in \widetilde{R}_i$ and $\rho > 0$ we have

$$\left(\sum_{j \neq i} \left[\frac{p_j(\mathbf{r})}{p_i(\mathbf{r})}\right]^{\lambda}\right)^{\rho} \geq 1.$$

For $\mathbf{r} \notin \widetilde{R}_i$ and $\rho > 0$ we have

$$\left(\sum_{j \neq i} \left[\frac{p_j(\mathbf{r})}{p_i(\mathbf{r})}\right]^{\lambda}\right)^{\rho} \geq 0.$$

Thus

$$I[\widetilde{R}_i] \leq \left(\sum_{j \neq i} \left[\frac{p_j(\mathbf{r})}{p_i(\mathbf{r})}\right]^{\lambda}\right)^{\rho}.$$

Applying this bound to the expression for the error probability we obtain

$$P_{e,i} \leq \int_{R^N} \left(\sum_{j \neq i} \left[\frac{p_j(\mathbf{r})}{p_i(\mathbf{r})}\right]^{\lambda}\right)^{\rho} p_i(\mathbf{r})d\mathbf{r}$$
$$= \int_{R^N} [p_i(\mathbf{r})]^{1-\lambda\rho} \left(\sum_{j \neq i} [p_j(\mathbf{r})]^{\lambda}\right)^{\rho} d\mathbf{r}$$

for $\rho > 0$ and $\lambda > 0$. If we let $\lambda = \frac{1}{1+\rho}$ (this is the value that minimizes the bound, see

Gallager problem 5.6) the resulting bound is known as the Gallager bound.

$$P_{e,i} \le \int_{R^N} [p_i(\mathbf{r})]^{\frac{1}{1+\rho}} \left(\sum_{j \neq i} [p_j(\mathbf{r})]^{\frac{1}{1+\rho}} \right)^{\rho} d\mathbf{r}.$$

If we let $\rho = 1$ we obtain what is known as the Bhattacharayya bound.

$$\begin{aligned} P_{e,i} &\le \int_{R^N} [p_i(\mathbf{r})]^{\frac{1}{2}} \left(\sum_{j \neq i} [p_j(\mathbf{r})]^{\frac{1}{2}} \right) d\mathbf{r} \\ &= \sum_{j \neq i} \int_{R^M} \sqrt{p_i(\mathbf{r}) p_j(\mathbf{r})} d\mathbf{r}. \end{aligned}$$

The average error probability is then written as

$$P_e = \sum_{i=1}^{M} \pi_i P_{e,i}.$$

Example of Gallager bound for M-ary orthogonal signals in AWGN.

$$p_i(\mathbf{r}) = \frac{1}{\sqrt{\pi N_0}} e^{-(r_i - \sqrt{E})^2)/N_0} \prod_{j \neq i} \frac{1}{\sqrt{\pi N_0}} e^{-r_j^2/N_0}$$

$$\begin{aligned} P_{e,i} &\le \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{\pi N_0}} e^{-(r_i - \sqrt{E})^2/N_0} \prod_{j \neq i} \frac{1}{\sqrt{\pi N_0}} e^{-r_j^2/N_0} \right]^{\frac{1}{1+\rho}} \\ &\quad \left\{ \sum_{j \neq i} \left[\frac{1}{\sqrt{\pi N_0}} e^{-(r_j - \sqrt{E})^2/N_0} \prod_{k \neq j} \frac{1}{\sqrt{\pi N_0}} e^{-r_k^2/N_0} \right]^{\frac{1}{1+\rho}} \right\}^{\rho} d\mathbf{r} \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\left(\prod_{j=0}^{M-1} \frac{1}{\sqrt{\pi N_0}} e^{-r_j^2/N_0} \right) e^{2\sqrt{E} r_i/N_0} e^{-E/N_0} \right]^{\frac{1}{1+\rho}} \\ &\quad \left\{ \sum_{j \neq i} \left[\left(\prod_{k=0}^{M-1} \frac{1}{\sqrt{\pi N_0}} e^{-r_k^2/N_0} \right) e^{2\sqrt{E} r_j/N_0} e^{-E/N_0} \right]^{\frac{1}{1+\rho}} \right\}^{\rho} d\mathbf{r} \end{aligned}$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\left(\prod_{j=0}^{M-1} \frac{1}{\sqrt{\pi N_0}} e^{-r_j^2/N_0} \right) e^{-E/N_0} \right] \\ &\quad \left[\exp\{ \frac{2\sqrt{E} r_i}{(1+\rho) N_0} \} \right] \left\{ \sum_{j \neq i} \exp\{ \frac{2\sqrt{E} r_j}{(1+\rho) N_0} \} \right\}^{\rho} d\mathbf{r}. \end{aligned}$$

Let

$$g(z) = \exp\{ \sqrt{\frac{2E}{N_0}} \frac{z}{1+\rho} \}$$

where $z_i = r_i / \sqrt{N_0/2}$. Then

$$\begin{aligned} P_{e,i} &\le \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=0}^{M-1} \frac{e^{-z_j^2/2}}{\sqrt{2\pi}} e^{-E/N_0} g(z_i) \left[\sum_{j \neq i} g(z_j) \right]^{\rho} d\underline{z} \\ &= e^{-E/N_0} E[g(z_i)] E \left[\sum_{j \neq i} g(z_j) \right]^{\rho}. \end{aligned}$$

Now it is easy to show (by completing the square) that

$$E[g(z)] = \int_{-\infty}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}} \exp\{ \sqrt{\frac{2E}{N_0}} \frac{z}{1+\rho} \} dz$$

$$= \exp\left(\frac{E}{N_0 (1+\rho)^2} \right).$$

Let $f(x) = x^{\rho}$ where $0 \le \rho \le 1$. Then $f(x)$ is a concave function and thus by Jensen's inequality we have that

$$E[f(X)] \le f(E[X]).$$

Thus

$$\begin{aligned} E\left(\left[\sum_{j \neq i} g(z_j) \right]^{\rho} \right) &\le \left(E\left[\sum_{j \neq i} g(z_j) \right] \right)^{\rho} \\ &= \left(\sum_{j \neq i} E[g(z_j)] \right)^{\rho} \\ &= (M-1)^{\rho} (E[g(z_j)])^{\rho}. \end{aligned}$$

Thus

$$\begin{aligned} P_{e,i} &\le (M-1)^{\rho} e^{-E/N_0} (E[g(z)])^{1+\rho} \\ &= (M-1)^{\rho} \exp\{ -\frac{E}{N_0} + (1+\rho) \frac{E}{N_0 (1+\rho)^2} \} \\ &\le \exp\{ -\frac{E}{N_0} \left(\frac{\rho}{1+\rho} \right) + \rho \ln M \}. \end{aligned}$$

Now we would like to minimize the bound over the parameter ρ keeping in mind that the bound is only valid for $0 \le \rho \le 1$. Let $a = \frac{E}{N_0}$ and $b = \ln M$ and

$$f(\rho) = -a\frac{\rho}{1+\rho} + \rho b.$$

Then

$$f'(\rho) = 0 \Rightarrow \rho = \sqrt{\frac{a}{b}} - 1.$$

Since $0 \le \rho \le 1$ the minimum occurs at an interior point of the interval $[0,1]$ if

$$1/4 < \frac{\ln M}{\frac{E}{N_0}} < 1$$

in which case the bound becomes

$$P_{e,i} \le \exp\left\{ -\left(\sqrt{\frac{E}{N_0}} - \sqrt{\ln M} \right)^2 \right\}.$$

If $\ln M / (\frac{E}{N_0}) \le 1/4$ then $\rho_{\min} = 1$ in which case the upper bound becomes $P_{e,i} \le \exp\{\ln M - \frac{E}{2N_0}\}$. If $\frac{\ln M}{\frac{E}{N_0}} \ge 1$ then $\rho_{\min} = 0$ in which case the upper bound becomes

$P_{e,i} \le 1$. In summary the Gallager bound for M orthogonal signals in white Gaussian noise is

$$P_{e,i} \le \begin{cases} 1, & \frac{E}{N_0} \le \ln M \\ \exp\left\{ -\left(\sqrt{\frac{E}{N_0}} - \sqrt{\ln M} \right)^2 \right\}, & \ln M \le \frac{E}{N_0} \le 4 \ln M \\ \exp\{ -(\frac{E}{2N_0} - \ln M) \}, & \frac{E}{N_0} \ge 4 \ln M. \end{cases}$$

Normally a communication engineer is more concerned with the energy transmitted per bit rather than the energy transmitted per signal, E. If we let E_b be the energy transmitted per bit then these are related as follows

$$E_b = \frac{E}{\log_2 M}.$$

Thus the bound on the error probability can be expressed in terms of the energy transmitted per bit as

$$P_{e,i} \le \begin{cases} 1, & \frac{E_b}{N_0} \le \ln 2 \\ \exp_2\left\{ -\log_2 M \left(\sqrt{\frac{E_b}{N_0}} - \sqrt{\ln 2} \right)^2 \right\}, & \ln 2 \le \frac{E_b}{N_0} \le 4 \ln 2 \\ \exp_2\left\{ -\log_2 M \left(\frac{E_b}{2N_0} - \ln 2 \right) \right\}, & \frac{E_b}{N_0} \ge 4 \ln 2 \end{cases}$$

where $\exp_2\{x\}$ denotes 2^x. Note that as $M \to \infty$, $P_e \to 0$ if $\frac{E_b}{N_0} > \ln 2$ = -1.59dB. Below we plot the exact error probability and the Gallager bound for M orthogonal signals for $M = 8, 64, 512$.

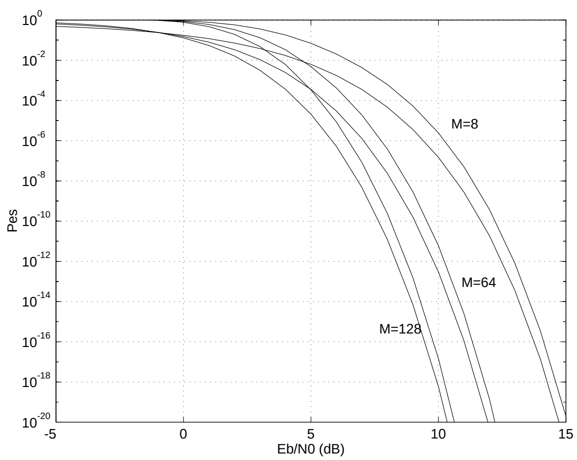


Figure 10: Comparison of Gallager Bound and exact error probability for orthogonal signals. In each group the upper signal is the bound and the lower signal is the exact error probability

Bit error probability

So far we have examined the symbol error probability for orthogonal signals. Usually the number of such signals is a power of 2, e.g. 4, 8, 16, 32, ... If so then each transmission of a signal is carrying $\log_2 M$ bits of information. In this case a communication engineer is usually interested in the bit error probability as opposed to the symbol error probability. Let $d(s_i, s_j)$ be the (Euclidean) distance between s_i and s_j, i.e

$$d^2(s_i, s_j) \triangleq \int_{-\infty}^{\infty} (s_i(t) - s_j(t))^2 dt = \sum_{l=1}^{\infty} |s_{i,l} - s_{j,l}|^2.$$

Now consider any signal set for which the distance between every pair of signals is the same. Orthogonal signal sets with equal energy satisfy this condition. Let $k = \log_2 M$. If s_i is transmitted there are $M-1$ other signals to which an error can be made. The number of signals which cause an error of i bits out of the k is $\binom{k}{i}$. Since all signals are the same distance from s_i the conditional probability of a symbol error causing i bits to be in error is

$$\frac{\binom{k}{i}}{(M-1)}.$$

So the average number of bit error given a symbol error is

$$\sum_{i=1}^{k} i \binom{k}{i} / (M-1) = \frac{k2^{k-1}}{M-1}.$$

So the probability of bit error given symbol error is

$$\frac{1}{k}\frac{k2^{k-1}}{M-1} = \frac{2^{k-1}}{2^k-1}.$$

So

$$P_{b,i} = \frac{2^{k-1}}{2^k-1} P_{e,i}$$

and this is true for any equidistant, equienergy signal set.

Union Bound

Assume

$$\pi_i = \frac{1}{M}, \quad 0 \le i \le M-1.$$

Let

$$\begin{aligned}
R_i &= \{\underline{r} : p_i(\underline{r}) > p_j(r) \quad \text{for all} \quad j \ne i\}, \\
\overline{R}_i &= \{\underline{r} : p_i(\underline{r}) \le p_j(\underline{r}) \quad \text{for some} \quad j \ne i\}, \\
&= \bigcup_{j=0, j\ne i}^{M-1} \{\underline{r} : p_i(\underline{r}) \le p_j(\underline{r})\}, \\
\overline{R}_{ij} &= \{\underline{r} : p_i(\underline{r}) \le p_j(\underline{r})\}.
\end{aligned}$$

Then

$$\begin{aligned}
P_{e,i} &= P(\underline{r} \in \overline{R}_i \mid H_i) \\
&= P(\underline{r} \in \bigcup_{j\ne i} \overline{R}_{ij} \mid H_i) \\
&\le \sum_{j \ne i} P(\overline{R}_{ij} \mid H_i)
\end{aligned}$$

where

$$P(\overline{R}_{ij} \mid H_i) = P\left\{\frac{p_i(\underline{r})}{p_j(\underline{r})} \le 1 \mid H_i\right\}$$

This is the union bound.

We now consider the bound for an arbitrary signal set in additive white Gaussian noise.

Let

$$s_i(t) = \sum_{l=0}^{N-1} s_{il}\varphi_l(t), \quad 0 \le i \le M-1.$$

For additive white Gaussian noise

$$p_i(\underline{r}) = \prod_{l=0}^{N-1} \left(\frac{\exp\{-\frac{1}{N_0}(r_l - s_{il})^2\}}{\sqrt{\pi N_0}}\right)$$

$$\begin{aligned}
\frac{p_i(\underline{r})}{p_j(\underline{r})} &= \prod_{l=0}^{N-1} \exp\{-\frac{1}{N_0}[(r_l - s_{il})^2 - (r_l - s_{jl})^2]\} \\
&= \exp\left\{\frac{2}{N_0}(\underline{r}, \underline{s}_i - \underline{s}_j) + \frac{E_j - E_i}{N_0}\right\}
\end{aligned}$$

where $(\underline{r}, \underline{s}_i - \underline{s}_j) = \sum_{l=0}^{N-1} r_l(s_{il} - s_{jl})$ and $E_k = \sum_{l=0}^{N-1} s_{kl}^2$ for $0 \le k \le M-1$. Thus

$$P(\overline{R}_{ij} \mid H_i) = P\left\{\frac{2}{N_0}(\underline{r}, \underline{s}_i - \underline{s}_j) \le \frac{E_i - E_j}{N_0} \mid H_i\right\}.$$

To do this calculation we need to calculate the statistics of the random variable $(r, s_i - s_j)$. The mean and variance are calculated as follows.

$$\begin{aligned}
E[(\underline{r}, \underline{s}_i - \underline{s}_j) \mid H_i] &= E[(\underline{n} + \underline{s}_i, \underline{s}_i - \underline{s}_j] \\
&= E_i - (\underline{s}_i, \underline{s}_j). \\
\text{Var}[(\underline{r}, \underline{s}_i - \underline{s}_j) \mid H_i] &= \text{Var}\ [(\underline{n} + \underline{s}_i, s_i - s_j)] \\
&= \frac{N_0}{2} \parallel \underline{s}_i - \underline{s}_j \parallel^2.
\end{aligned}$$

Also $(r, \underline{s}_i - \underline{s}_j)$ is a Gaussian random variable. Thus

$$P\left\{(\underline{r}, \underline{s}_i - \underline{s}_j) \le \frac{E_i - E_j}{2}\right\} = \Phi\left(\frac{\frac{E_i - E_j}{2} - (E_i - (s_i, s_j))}{\sqrt{\frac{N_0}{2}} \|\underline{s}_i - \underline{s}_j\|}\right)$$

$$
\begin{aligned}
&= Q\Big(\frac{E_i - 2(s_i,s_j) + E_j}{\sqrt{2N_0}\;\|\underline{s}_i - \underline{s}_j\|}\Big) \\
&= Q\Big(\frac{\|\underline{s}_i - \underline{s}_j\|}{\sqrt{2N_0}}\Big).
\end{aligned}
$$

Thus the union bound on the error probability is given as

$$P_{e,i} \le \sum_{j \ne i} Q\Big(\frac{\|\underline{s}_j - \underline{s}_i\|}{\sqrt{2N_0}}\Big).$$

Note that $\|\underline{s}_i - \underline{s}_j\|^2 = d_E^2(s_i, s_j)$, i.e. the square of the Euclidean distance.

Example: Hamming Code

The Hamming code has the following set of codewords.

$$
\begin{array}{ll}
(0,0,0,0,0,0,0) & (1,1,1,1,1,1,1) \\
(1,0,0,0,1,0,1) & (0,1,0,0,1,1,1) \\
(1,1,0,0,0,1,0) & (1,0,1,0,0,1,1) \\
(0,1,1,0,0,0,1) & (1,1,0,1,0,0,1) \\
(1,0,1,1,0,0,0) & (1,1,1,0,1,0,0) \\
(0,1,0,1,1,0,0) & (0,1,1,1,0,1,0) \\
(0,0,1,0,1,1,0) & (0,0,1,1,1,0,1) \\
(0,0,0,1,0,1,1) & (1,0,0,1,1,1,0)
\end{array}
$$

These are mapped to signals using the mapping

$$
\begin{aligned}
0 \rightarrow &+ \sqrt{E} \\
1 \rightarrow &- \sqrt{E}
\end{aligned}
$$

The 16 signals are

$$
\begin{array}{rl}
\sqrt{E}(1,1,1,1,1,1,1) & \sqrt{E}(-1,-1,-1,-1,-1,-1,-1) \\
\sqrt{E}(-1,1,1,1,-1,1,-1) & \sqrt{E}(1,-1,1,1,-1,-1,-1) \\
\sqrt{E}(-1,-1,1,1,1,-1,1) & \sqrt{E}(-1,1,-1,1,1,-1,-1) \\
\sqrt{E}(1,-1,-1,1,1,1,-1) & \sqrt{E}(-1,-1,1,-1,1,1,-1) \\
\sqrt{E}(-1,1,-1,-1,1,1,1) & \sqrt{E}(-1,-1,-1,1,-1,1,1) \\
\sqrt{E}(1,-1,1,-1,-1,1,1) & \sqrt{E}(1,-1,-1,-1,1,-1,1) \\
\sqrt{E}(1,1,-1,1,-1,-1,1) & \sqrt{E}(1,1,-1,-1,-1,1,-1) \\
\sqrt{E}(1,1,1,-1,1,-1,-1) & \sqrt{E}(-1,1,1,-1,-1,-1,1)
\end{array}
$$

Let $c_0, \ldots, c_{15}$ be the 16 binary codewords with code symbols being either 0 or 1. Let $s_0, \ldots, s_{15}$ be the 16 signals with coefficients being either $-\sqrt{E}$ or $+\sqrt{E}$.

Note that $d_E^2(s_i,s_j) = 4Ed_H(c_i,c_j)$ where $d_E(s_i,s_j)$ is the Euclidean distance between signal s_i and s_j and $d_H(c_i,c_j)$ is the Hamming distance between codeword c_i and c_j. Notice also that the signals are *geometrically uniform*. That is, the number of signals at a Hamming distance $4El$ from s_0 is the same as the number of signals at Hamming distance $4El$ from any other codeword.

$d_E^2(s_0,s_i)$	A_i
$3\times 4E$	7
$4\times 4E$	7
$7\times 4E$	1

Thus the union bound on the codeword error probability for the Hamming code with BPSK modulation (which yields a 16-ary signal set in 7 dimensions) is

$$P_e \leq 7Q(\sqrt{\frac{12E}{2N_0}}) + 7Q(\sqrt{\frac{16E}{2N_0}}) + 1Q(\sqrt{\frac{28E}{2N_0}})$$

For the Hamming code the energy per dimension E is related to the energy per information bit

E_b by

$$E = \frac{4}{7}E_b$$

Thus the error probabilty is bounded by

$$P_e \leq 7Q(\sqrt{\frac{2E_b(4/7)3}{N_0}}) + 7Q(\sqrt{\frac{2E_b(4/7)4}{N_0}}) + 1Q(\sqrt{\frac{2E_b(4/7)7}{N_0}})$$

Bit error probability for Hamming Code

In the Hamming code the shown above, the first four bits represent the information transmitted. From the code we can calculate the the number $A_{d,m}$ of codewords at a distance l with a m information bits that are one.

d	m	$A_{l,m}$
3	1	3
3	2	3
3	3	1
4	1	1
4	2	3
4	3	3
7	4	1

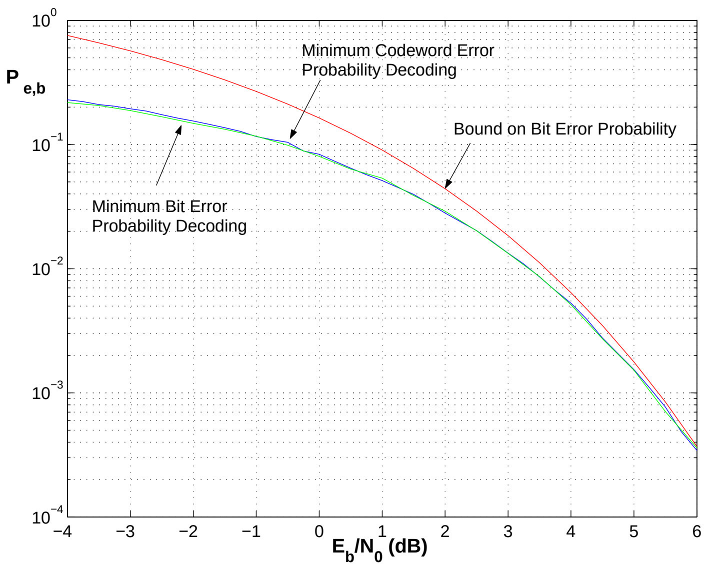
The bit error probability is then union-bounded by

$$\begin{aligned}
P_b &\leq \frac{1}{4}\sum_{d=3}^{7}\sum_{m=1}^{4} mA_{d,m}Q\left(\sqrt{\frac{2E_brd}{N_0}}\right)\\
&= \frac{1}{4}\sum_{d=3}^{7} Q\left(\sqrt{\frac{2E_brd}{N_0}}\right)\sum_{m=1}^{4} mA_{d,m}\\
&= \frac{1}{4}\sum_{d=3}^{7} B_dQ\left(\sqrt{\frac{2E_brd}{N_0}}\right)\\
&= \frac{1}{4}\left[12Q\left(\sqrt{\frac{2E_br3}{N_0}}\right) + 16Q\left(\sqrt{\frac{2E_br4}{N_0}}\right) + 4Q\left(\sqrt{\frac{2E_br7}{N_0}}\right)\right]\\
&= 3Q\left(\sqrt{\frac{2E_br3}{N_0}}\right) + 4Q\left(\sqrt{\frac{2E_br4}{N_0}}\right) + Q\left(\sqrt{\frac{2E_br7}{N_0}}\right)
\end{aligned}$$

where

$$B_d = \sum_{m=1}^{4} mA_{d,m}$$

and $r = 4/7$ is the rate of the code.



Union-Bhattacharyya Bound

We now use the following to derive the Union-Bhattacharyya bound. This is an alternate way of obtaining this bound. We could have started with the Union-Bhattacharyya bound derived from the Gallager bound, but we would get the same answer.

Fact: $Q(x) \leq \frac{1}{2}e^{-x^2/2} \leq e^{-x^2/2}, \quad x \geq 0$. (To prove this let X_1 and X_2 be independent Gaussian random variables mean 0 variance 1. Then show $Q^2(x) = P(X_1 \geq x, X_2 \geq x) \leq \frac{1}{4}P(X_1^2 + X_2^2 \geq \sqrt{2}\,x)$. Use the fact the $X_1^2 + X_2^2$ has Rayleigh density; see page 29 of Proakis)

Using this fact leads to the bound

$$P_{e,i} \leq \sum_{j \neq i} \exp\left\{-\frac{\|\underline{s}_i - \underline{s}_j\|^2}{4N_0}\right\}.$$

This is the Union Bhattacharyya bound for an additive white Gaussian noise channel.

Random Coding

Now consider 2^{NM} communication systems corresponding to all possible signals where

$$s_{ij} = \pm\sqrt{E}$$
$$\|s_i\|^2 = NE \qquad 0 \leq i \leq M-1$$

Consider the average error probability, averaged over all possible selections of signal sets

For example: Let $N = 3$, $M = 2 \Rightarrow$ there are $2^{3\times 2} = 2^6 = 64$ possible sets of 2 signals with each signal a linear combination of three orthogonal signals with the coefficients required to be one of two values.

Set number 1	$s_0(t)$	$=$	$-\sqrt{E}\,\varphi_1(t) - \sqrt{E}\,\varphi_2(t) - \sqrt{E}\,\varphi_3(t)$
	$s_1(t)$	$=$	$-\sqrt{E}\,\varphi_1(t) - \sqrt{E}\,\varphi_2(t) - \sqrt{E}\,\varphi_3(t)$
Set number 2	$s_0(t)$	$=$	$-\sqrt{E}\,\varphi_1(t) - \sqrt{E}\,\varphi_2(t) - \sqrt{E}\,\varphi_3(t)$
	$s_1(t)$	$=$	$-\sqrt{E}\,\varphi_1(t) - \sqrt{E}\,\varphi_2(t) + \sqrt{E}\,\varphi_3(t)$
Set number 3	$s_0(t)$	$=$	$-\sqrt{E}\,\varphi_1(t) - \sqrt{E}\,\varphi_2(t) + \sqrt{E}\,\varphi_3(t)$
	$s_1(t)$	$=$	$-\sqrt{E}\,\varphi_1(t) - \sqrt{E}\,\varphi_2(t) - \sqrt{E}\,\varphi_3(t)$
Set number 4	$s_0(t)$	$=$	$-\sqrt{E}\,\varphi_1(t) - \sqrt{E}\,\varphi_2(t) + \sqrt{E}\,\varphi_3(t)$
	$s_1(t)$	$=$	$-\sqrt{E}\,\varphi_1(t) - \sqrt{E}\,\varphi_2(t) + \sqrt{E}\,\varphi_3(t)$
Set number 5	$s_0(t)$	$=$	$-\sqrt{E}\,\varphi_1(t) - \sqrt{E}\,\varphi_2(t) - \sqrt{E}\,\varphi_3(t)$
	$s_1(t)$	$=$	$-\sqrt{E}\,\varphi_1(t) + \sqrt{E}\,\varphi_2(t) - \sqrt{E}\,\varphi_3(t)$
$\vdots$	$\vdots$		
Set number 64	$s_0(t)$	$=$	$+\sqrt{E}\,\varphi_1(t) + \sqrt{E}\,\varphi_2(t) + \sqrt{E}\,\varphi_3(t)$
	$s_1(t)$	$=$	$+\sqrt{E}\,\varphi_1(t) + \sqrt{E}\,\varphi_2(t) + \sqrt{E}\,\varphi_3(t)$

Let $P_{e,i}(k)$ be the error probability of signal set k given H_i. Then

$$\overline{P}_{e,i} = \frac{1}{2^{NM}} \sum_{k=1}^{2^{NM}} P_{e,i}(k)$$

and

$$\overline{P}_e = \frac{1}{M} \sum_{i=1}^{M} \overline{P}_{e,i}$$

and

$$P_e(k) = \frac{1}{M} \sum_{i=1}^{M} P_{e,i}(k).$$

If $\overline{P}_e \leq \alpha$ then at least one of the of the 2^{NM} signals sets must have $P_e(k) \leq \alpha$ (otherwise $P_e(k) > \alpha$ for all $k \Rightarrow \overline{P}_e > \alpha$; contradiction). In other words there exists a signal set with $P_e \leq \alpha$. This is known as the random coding argument. Let $s_{i,j},\ 0 \leq i \leq M-1,\ 1 \leq j \leq N$ be independent identically distributed random variables with $P(s_{i,j} = +\sqrt{E}) = P(s_{i,j} = -\sqrt{E}) = \frac{1}{2}$ and $\overline{P}_e = E[P_e(s)]$ where the expectation is with respect to the random variables s.

$$\overline{P}_{e,i} = E[P_{e,i}(s)] \leq \sum_{j \neq i} E\left[\exp\ \left\{-\frac{||s_i - s_j||^2}{4N_0}\right\}\right].$$

Let $X_{ij} = ||s_i - s_j||^2 = \sum_{l=1}^{N} (s_{il} - s_{jl})^2$. Then

$$\begin{aligned} P(X_{ij} = 4Em) &= P(s_i \text{ and } s_j \text{ differ in } m \text{ places out of } N) \\ &= \binom{N}{m} 2^{-N} \end{aligned}$$

since $P(s_{il} = s_{jl}) = P(s_{il} \neq s_{jl}) = \frac{1}{2}$.

So

$$E[\exp\ \left\{-\frac{||s_i - s_j||^2}{4N_0}\right\}] = E[e^{-X_{ij}/4N_0}]$$

$$\begin{aligned} E[e^{-X_{ij}/4N_0}] &= \sum_{m=0}^{N} \binom{N}{m} 2^{-N} e^{-m4E/4N_0} \\ &= 2^{-N} \sum_{m=0}^{N} \binom{N}{m} (e^{-E/N_0})^m \\ &= 2^{-N} (1 + e^{-E/N_0})^N \\ &= \exp_2[-N\left(1 - \log_2(1 + e^{-E/N_0})\right)] \end{aligned}$$

Let $R_0 = 1 - \log_2\ (1 + e^{-E/N_0})$ then

$$\overline{P}_{e,i} \leq \sum_{j \neq i} 2^{-NR_0} = (M-1)2^{-NR_0} \leq M 2^{-NR_0} = 2^{-N(R_0 - R)}$$

where $R = \frac{\log_2 M}{N}$ is the number of bits transmitted per dimension and E is the signal energy per dimension. We have shown that there exist a signal set for which the average value of the error probability for the $i-th$ signal is small. Thus we have shown that as N goes to ∞ the error probability given s_i was transmitted goes to zero if the rate is less than the cutoff rate R_0. This however does not imply that there exist a code $s_0, \ldots, s_{M-1}$ such that $P_{e,0}, \ldots, P_{e,M-1}$ are simultaneously small. It is possible that $P_{e,i}$ is small for some code for which $P_{e,j}$ is large. We now show that we can simultaneously make each of the error probabilities small simultaneously. First choose a code with $M = (2)2^{RN}$ codewords for which the average error probability is less than say $\varepsilon_N/2$ for large N. If more than 2^{NR} of these codewords has $P_{e,i} \geq \varepsilon_N$ then the average error probability would be greater than $\varepsilon_N/2$, a contradiction. Thus at least $M/2 = 2^{NR}$ of the codewords must have $P_{e,i} \leq \varepsilon_N$. So delete the codewords that have $P_{e,i} \geq \varepsilon_N$ (less than half). We obtain a code with (at least) 2^{NR} codewords with $P_{e,i} \to 0$ as $n \to \infty$ for $R < R_0$.

Thus we have proved the following.

Theorem: There exist a signal set with M signals in N dimensions with $P_e \leq 2^{-N(R_0 - R)}$ ($\Rightarrow P_e \to 0$ *as* $N \to \infty$ *provided* $R < R_0$).

Note: E is the energy per dimension. Each signal then has energy NE and is transmitting $\log_2 M$ bits of information so that $E_b = \frac{NE}{\log_2 M} = E/R$ is the energy per bit of information.

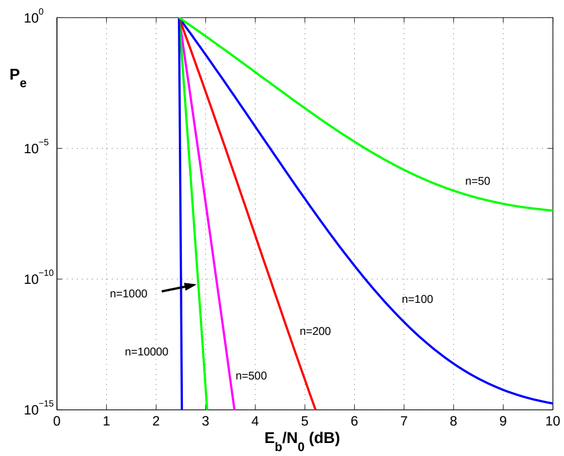


Figure 11: Error Probabilities Based on Cutoff Rate for Binary Input-Continuous Output Channel for Rate 1/2 codes

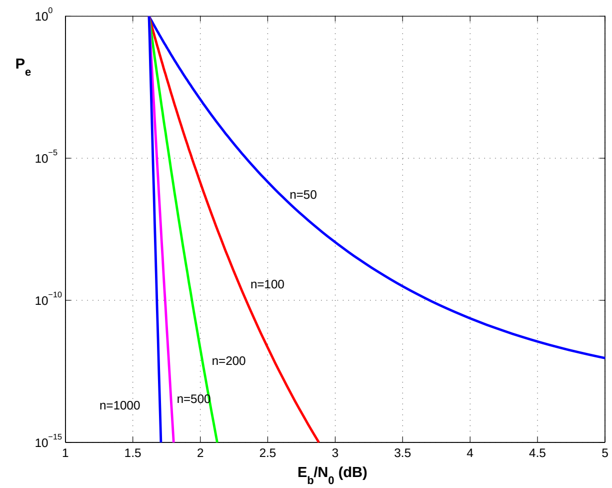


Figure 12: Error Probabilities Based on Cutoff Rate for Binary Input-Continuous Output Channel for Rate 1/8 codes

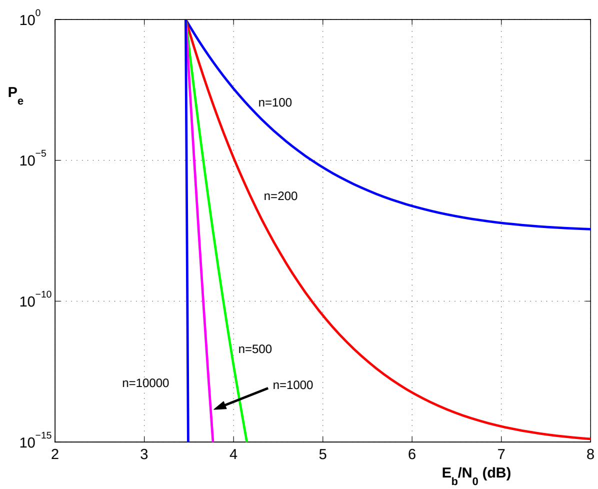


Figure 13: Error Probabilities Based on Cutoff Rate for Binary Input-Continuous Output Channel for Rate 3/4 codes

From the theorem, reliable communication ($P_e \to 0$) is possible provided $R < R_0 \leq 1$, i.e.

$$1 - \log_2 \left(1 + \exp\left\{-E_b R/N_0\right\}\right) > R$$

$$1 - R > \log_2(1 + e^{-E_b R/N_0})$$

$$2^{1-R} > 1 + e^{-E_b R/N_0} \Rightarrow e^{-E_b R/N_0} < 2^{1-R} - 1$$

$$-E_b R/N_0 < \ln(2^{1-R} - 1) \Rightarrow \frac{E_b}{N_0} \geq -\frac{\ln(2^{1-R} - 1)}{R}$$

For

$$R \to 0 \qquad -\frac{\ln(2^{1-R} - 1)}{R} \to 2\ln 2 \Rightarrow P_e \to 0 \text{ if } E_b/N_0 > 2\ln 2$$

Note: M orthogonal signals have $P_e \to 0$ if $E_b/N_0 > \ln 2$. The rate of orthogonal signals is

$$R = \frac{\log_2 M}{N} = \frac{\log_2 M}{M} \to 0 \text{ as } M \to \infty$$

The theorem guarantees existence of signals with $\frac{\log_2 M}{N} = R > 0$ and $P_e \to 0$ as $M \to \infty$.

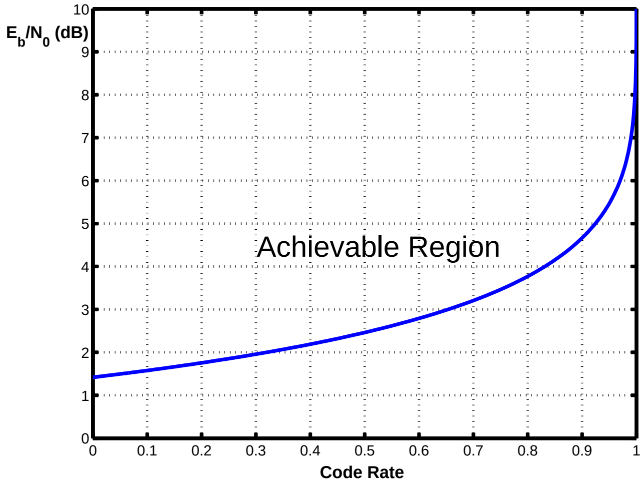


Figure 14: Cutoff Rate for Binary Input-Continuous Output Channel

Example of Gallager bound for M-ary signals in AWGN.

Now we evaluate the Gallager bound for an arbitrary signal set in additive white Gaussian noise channel. As usual assume the signal set transmitted has the form

$$s_i(t) = \sum_{j=0}^{N-1} \mu_{i,j}\phi_j(t), \quad i = 0,1,\ldots,M-1$$

The optimal receiver does a correlation with each of the orthonormal functions to produce the decision statistic $r_0,\ldots,r_{N-1}$. The conditional density function of r_i given signal $s_i(t)$ transmitted is given by

$$p_i(\mathbf{r}) = \prod_{k=1}^{N} \frac{1}{\sqrt{\pi N_0}} e^{-(r_k-\mu_{i,k})^2/N_0}$$

If we substitute this into the general form of the Gallager bound we obtain

$$\begin{aligned} P_{e,i} \;\le\; & \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\prod_{k=1}^{N}\frac{1}{\sqrt{\pi N_0}}e^{-(r_k-\mu_{i,k})^2/N_0}\right]^{\frac{1}{1+\rho}} \\ & \left\{\sum_{j\neq i}\left[\prod_{k=1}^{N}\frac{1}{\sqrt{\pi N_0}}e^{-(r_k-\mu_{j,k})^2/N_0}\right]^{\frac{1}{1+\rho}}\right\}^{\rho} d\mathbf{r} \end{aligned}$$

$$\begin{aligned} &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\left(\prod_{k=1}^{N}\frac{1}{\sqrt{\pi N_0}}e^{-(r_k^2-2r_k\mu_{i,k}+\mu_{i,k}^2)/N_0(1+\rho)}\right)\right] \\ &\qquad \left\{\sum_{j\neq i}\left[\prod_{k=1}^{N}e^{-(r_k^2-2r_k\mu_{j,k}+\mu_{j,k}^2)/N_0}\right]^{\frac{1}{1+\rho}}\right\}^{\rho} d\mathbf{r} \\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{k=1}^{N}\frac{1}{\sqrt{\pi N_0}}\exp\{-\frac{(r_k^2-2r_k\mu_{i,k}+\mu_{i,k}^2)}{N_0(1+\rho)}-\frac{(r_k)^2\rho}{N_0(1+\rho)}\} \\ &\qquad \left\{\sum_{j\neq i}\prod_{k=1}^{N}\exp\{-\frac{(-2r_k\mu_{j,k}+\mu_{j,k}^2)}{N_0(1+\rho)}\}\right\}^{\rho} d\mathbf{r} \\ &= \exp\{-\frac{\|\mu_i\|^2\rho}{N_0(1+\rho)^2}\}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{k=1}^{N}\frac{1}{\sqrt{\pi N_0}}\exp\{-\frac{1}{N_0}(r_k-\frac{\mu_{i,k}}{1+\rho})^2\} \\ &\qquad \left\{\sum_{j\neq i}\exp\{-\frac{\|\mu_j\|^2}{N_0(1+\rho)}\}\prod_{k=1}^{N}\exp\{\frac{2r_k\mu_{j,k}}{N_0(1+\rho)}\}\right\}^{\rho} d\mathbf{r} \\ &\le \exp\{-\frac{\|\mu_i\|^2\rho}{N_0(1+\rho)^2}\}\left\{\sum_{j\neq i}\exp\{-\frac{\|\mu_j\|^2}{N_0(1+\rho)}\}\right. \end{aligned}$$

$$\begin{aligned} & \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\prod_{k=1}^{N}\frac{1}{\sqrt{\pi N_0}}\exp\{-\frac{1}{N_0}(r_k-\frac{\mu_{i,k}}{1+\rho})^2\}\right)\exp\{\frac{2r_k\mu_{j,k}}{N_0(1+\rho)}\}d\mathbf{r}\Bigg\}^{\rho} \\ = \; & \exp\{-\frac{\|\mu_i\|^2\rho}{N_0(1+\rho)^2}\}\left\{\sum_{j\neq i}\exp\{-\frac{\|\mu_j\|^2}{N_0(1+\rho)}\}\exp\{-\frac{1}{N_0}\left[\frac{\|\mu_i\|^2}{(1+\rho)^2}-\frac{\|\mu_i+\mu_j\|^2}{(1+\rho)^2}\right]\}\right\}^{\rho} \\ = \; & \left(\sum_{j\neq i}\exp\left\{-\frac{1}{N_0}\left[\frac{d_E^2(\mu_i,\mu_j)}{(1+\rho)^2}-\frac{(1-\rho)\|\mu_j\|^2}{(1+\rho)^2}\right]\right\}\right)^{\rho} \end{aligned}$$

When the signals are all orthogonal to each other then $d_E^2 = 2E$ for $i \neq j$ and $\|\mu_j\|^2 = E$ and the bound becomes

$$\begin{aligned} P_{e,i} &\le \left(\sum_{j\neq i}\exp\left\{-\frac{1}{N_0}\left[\frac{2E}{(1+\rho)^2}-\frac{(1-\rho)E}{(1+\rho)^2}\right]\right\}\right)^{\rho} \\ &= (M-1)^{\rho}\exp\left\{-\frac{E\rho}{N_0(1+\rho)}\right\} \end{aligned}$$

This is identical to the previous expression.

Now we consider a couple of different signal sets. The first signal set has 16 signals in seven dimensions. The energy in each dimension is E so the total energy transmitted is $7E$. The energy transmitted per information bit is $E_b = 7E/4$ The geometry of the signal set is such that for any signal there are seven other signals at Euclidean distance $12E$, seven other signals at Euclidean distance $16E$ and one other signal at distance $28E$. All signals have energy $7E$. (This is called the Hamming code). The fact that the signal set is geometrically uniform is due to the linearity of the code. We plot the Gallager bound for $\rho = 0.1, 0.2, \ldots, 1.0$. The Union-Bhattacharyya bound is the Gallger bound with $\rho = 1.0$. The second signal set has 256 signals in 16 dimensions with 112 signals at distance $24E$, 16 signals at distance $32E$, 112 signals at distance $40E$ and 15 signals at distance $64E$. In this case $E_b = 2E$.

As can be seen from the figures the union bound is the tightest bound except at very low signal-to-noise ratios where the Gallager bound stays below 1. At reasonable signal-to-noise ratios the optimum ρ in the Gallager bound is 1 and thus it reduces to the Union-Bhattacharyya bound.

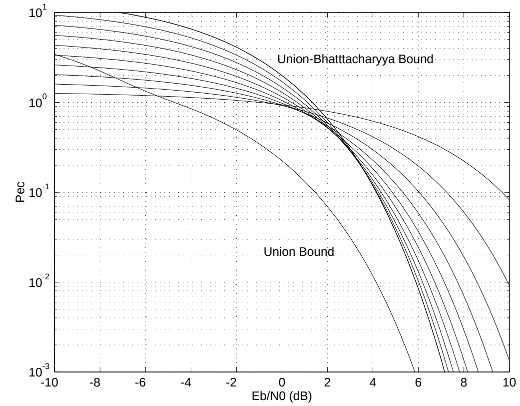


Figure 15: Comparison of Gallager Bound, Union Bound and Union Bhattacharyya Bound for the Hamming Code with BPSK Modulation

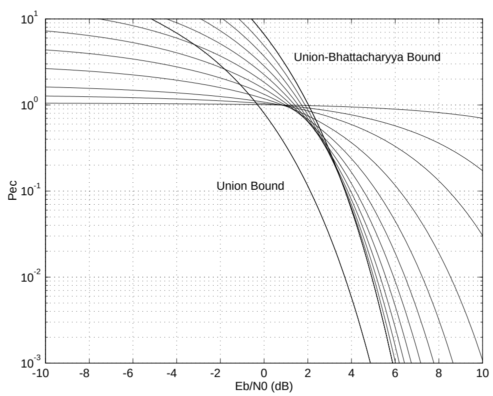


Figure 16: Comparison of Gallager Bound, Union Bound and Union Bhattacharyya Bound for the Nordstrom-Robinson code with BPSK Modulation