Lecture Notes 5: Noncoherent Receivers

Goals

- Derive optimum receiver for arbitrary signals in Gaussian noise with a random phase.
- Determine performance of two signals in white Gaussian noise.
- Determine performance of M-orthogonal signals in white Gaussian noise.

System Model

Assume additive stationary Gaussian noise and that \( s_k(t) \) for \( 0 \leq k \leq M - 1 \) has the form

\[
 s_k(t) = a_k(t) \cos(\omega_k t + \beta_k(t)), \quad k = 0, 1, 2, \ldots, M - 1
\]

with

\[
\int s_k^2(t) dt = \frac{1}{2} \int a_k^2(t) dt = E
\]

where \( a_k(t) \) and \( \beta_k(t) \) are lowpass waveforms with respect to \( \omega_k \). When \( s_k(t) \) is transmitted the received waveform has the form

\[
r(t) = a_k(t) \cos(\omega_k t + \beta_k(t) + \theta_k) + n(t)
\]

where \( \theta_k \) is a random phase. If \( \theta_k = 0 \) with probability 1 then we have the usual coherent reception situation already discussed. We will for this section assume that \( \theta_k \) is uniformly distributed on the interval \([0, 2\pi]\) and that the receiver does not know what \( \theta_k \) is.

We can use the representation of bandpass signals and noise in deriving the optimal receiver.

\[
r(t) = \text{Re}[u_k(t)e^{j\theta_k}] + n(t)
\]

where

\[
u_k(t) = a_k(t) \cos \beta_k(t) + j a_k(t) \sin \beta_k(t)
\]

and \( z_l \) is a complex Gaussian random variable with mean zero and with

\[
E[\text{Re}(z_l)^2] = \lambda_l
\]

\[
E[\text{Im}(z_l)^2] = \lambda_l
\]

\[
E[\text{Re}(z_l)\text{Im}(z_l)] = 0.
\]

We can now calculate the probability density of \( \tilde{r} = (\tilde{r}_1, \ldots, \tilde{r}_N) \) conditioned on the value of \( \theta_k \). Let \( \tilde{r}_l = u_k(t)e^{j\theta_k} + z_l \) then

\[
p_k(\tilde{r} | \theta_k) = \frac{1}{2\pi \lambda_k} \exp \left\{ -\frac{1}{2\lambda_k} \tilde{r}^* e^{-j\theta_k} u_k^* | \tilde{r}^* e^{j\theta_k} u_k |^2 \right\}
\]

Let \( p_k(\tilde{r} | \theta_k) = \frac{1}{2\pi \lambda_k} \exp \left\{ -\frac{1}{2\lambda_k} \tilde{r}^* e^{-j\theta_k} u_k^* | \tilde{r}^* e^{j\theta_k} u_k |^2 \right\} \),

\[
p_k(\tilde{r} | \theta_k) = \frac{1}{\prod_{l=1}^N 2\pi \lambda_l} \exp \left\{ -\frac{1}{2} \sum_{l=1}^N |\tilde{r}_l - e^{j\theta_k} u_k |^2 \right\}
\]

\[
p_k(\tilde{r} | \theta_k) = \frac{1}{\prod_{l=1}^N 2\pi \lambda_l} \exp \left\{ -\frac{1}{2} \sum_{l=1}^N |\tilde{r}_l - u_k |^2 + |u_k |^2 - 2 \text{Re}(\tilde{r}_l e^{-j\theta_k} u_k^*) \right\}
\]

\[
= \frac{1}{\prod_{l=1}^N 2\pi \lambda_l} \exp \left\{ -\frac{1}{2} \sum_{l=1}^N |\tilde{r}_l |^2 - 2 \text{Re}(\tilde{r}_l e^{-j\theta_k} u_k^*) \right\}
\]

and \( \tilde{r}_l \) is a complex Gaussian random variable with mean zero and with

\[
E[\text{Re}(\tilde{r}_l)] = \lambda_l
\]

\[
E[\text{Im}(\tilde{r}_l)] = \lambda_l
\]

\[
E[\text{Re}(\tilde{r}_l)\text{Im}(\tilde{r}_l)] = 0.
\]
\[
\begin{align*}
&= \prod_{i=1}^{N}(2\pi\lambda_i) \left\{ \exp \left\{ -\frac{1}{2} \left[ \sum_{i=1}^{N} |\tilde{\tau}_i|^2 + |u_k|^2 \right] \right\} + \sum_{i=1}^{N} \frac{\text{Re}(\tilde{\tau}_i u_k^*) \cos(\theta_k) - \text{Im}(\tilde{\tau}_i u_k^*) \sin(\theta_k)}{\lambda_i} \right\} \\
&= \prod_{i=1}^{N}(2\pi\lambda_i) \left\{ \exp \left\{ -\frac{1}{2} \left[ \sum_{i=1}^{N} |\tilde{\tau}_i|^2 + |u_k|^2 \right] \right\} + \left( \sum_{i=1}^{N} \frac{\text{Re}(\tilde{\tau}_i u_k^*)}{\lambda_i} \right) \cos(\theta_k) \right\}. \\
\end{align*}
\]

The joint density given signal \(k\) transmitted is then obtained by averaging with respect to \(\theta_k\). The joint density given signal \(k\) transmitted is then
\[
p_k(\tilde{\tau}_1, \ldots, \tilde{\tau}_N) = \int_{-\infty}^{\infty} \frac{1}{2\pi} p_k(\tilde{\tau}_1, \ldots, \tilde{\tau}_N|\theta_k) d\theta_k \\
= \prod_{i=1}^{N}(2\pi\lambda_i) \left\{ \exp \left\{ -\frac{1}{2} \left[ \sum_{i=1}^{N} |\tilde{\tau}_i|^2 + |u_k|^2 \right] \right\} + \left( \sum_{i=1}^{N} \frac{\text{Re}(\tilde{\tau}_i u_k^*)}{\lambda_i} \right) \} \right\} d\theta_k \\
= \prod_{i=1}^{N}(2\pi\lambda_i) \left\{ \exp \left\{ -\frac{1}{2} \left[ \sum_{i=1}^{N} |\tilde{\tau}_i|^2 + |u_k|^2 \right] \right\} \right\} \\
= \prod_{i=1}^{N}(2\pi\lambda_i) \left\{ \exp \left\{ -\frac{1}{2} \left[ \sum_{i=1}^{N} |\tilde{\tau}_i|^2 + |u_k|^2 \right] \right\} \right\} I_0 \left( \left( \sum_{i=1}^{N} \frac{\text{Re}(\tilde{\tau}_i u_k^*)}{\lambda_i} \right) \right),
\]
where
\[
I_0(x) = \frac{1}{2\pi} \int_{0}^{2\pi} \exp(x \cos \theta) \, d\theta.
\]

Similarly
\[
\sum_{i=1}^{N} \frac{\text{Re}(\tilde{\tau}_i u_k^*)}{\lambda_i} = \int \tilde{\tau}(t) q_k(t) dt.
\]
Thus the optimal receiver computes the following likelihood ratios
\[
\Lambda_{k,0}(\tilde{\tau}(t)) = \lim_{N \to \infty} (r(t)) = \exp \left\{ -\frac{1}{2} \int u_k(t) q_k(t) dt \right\} \int \left| \tilde{\tau}(t) q_k(t) dt \right|.
\]
and chooses \( k \) for which \( \Lambda_{k,0} \) is maximum.

**Special Case: White Gaussian Noise**

In this case the integral equation is easily solved:
\[
q_k(t) = \frac{2}{N_0} u(t)
\]
so that
\[
\Lambda_{k,0}(\tilde{\tau}(t)) = \exp \left\{ -\frac{1}{2} \int |u(t)|^2 dt \right\} I_0 \left( \left( \sum_{i=1}^{N} \frac{\text{Re}(\tilde{\tau}_i u_k^*)}{\lambda_i} \right) \right).
\]

For equi-energy signals this reduces to choosing \( k \) that maximizes
\[
I_0 \left( \left( \sum_{i=1}^{N} \frac{\text{Re}(\tilde{\tau}_i u_k^*)}{\lambda_i} \right) \right)
\]
and since \( I_0(x) \) is an increasing function of \( x \) the optimal receiver chooses \( k \) to maximize
\[
\int |\tilde{\tau}(t) q_k(t) dt|^2 = (\text{Re} \int \tilde{\tau}(t) u_k^*(t) dt)^2 + (\text{Im} \int \tilde{\tau}(t) u_k^*(t) dt)^2.
\]
Now note that
\[
r(t) = \text{Re} \left[ \tilde{\tau}(t) e^{i\omega_0 t} \right]
\]
and consider the following integral
\[
\int r(t) u_k(t) \cos(\omega_0 t + \beta_k) dt = \int \text{Re}[\tilde{\tau}(t)] e^{i\omega_0 t} \text{Re}[u_k(t)] e^{i\beta_k} dt.
\]
Since (can you show this) for any two complex numbers \( a \) and \( b \)
\[
\text{Re}[a] \text{Re}[b] = \frac{1}{2} \left[ \text{Re}[ab]^* + \frac{1}{2} \text{Re}[ab] \right]
\]
the above integral becomes
\[
\int r(t) u_k(t) \cos(\omega_0 t + \beta_k) dt = \frac{1}{2} \int \text{Re}[\tilde{\tau}(t)] u_k^*(t) dt + \frac{1}{2} \int \text{Re}[\tilde{\tau}(t)] u_k^*(t) e^{2i\omega_0 t} dt,
\]
That the second term is zero is due to the fact that both \( \tilde{\tau} \) and \( u_k \) are lowpass processes. Thus
\[
\int r(t) u_k(t) \cos(\omega_0 t + \beta_k) dt = \frac{1}{2} \int \text{Re}[\tilde{\tau}(t)] u_k^*(t) dt.
\]
Similarly
\[ \int r(t) a_k(t) \sin(\omega_k t + \beta_k(t)) dt = \frac{1}{2} \int \text{Im}[\tilde{r}(t)] a_k^*(r) dt, \]

Thus an equivalent form of the optimal receiver computes the following for each \( k \)

\[ (\int r(t) a_k(t) \cos(\omega_k t + \beta_k(t)) dt)^2 + (\int r(t) a_k(t) \sin(\omega_k t + \beta_k(t)) dt)^2 \]

and decides that signal \( k \) was transmitted if \( k \) maximizes the above expression.

**Performance in AWGN**

Consider a binary communication system with noncoherent reception. Assume the two transmitted signals are

\[ s_k(t) = a_k(t) \cos(\omega_k t + \beta_k(t)), \quad k = 1, 2 \]

with

\[ \frac{1}{2E} \int s_0(t)s_1(t) dt = \rho_{01} = \rho \]

Let \( H_1 \) denote the event that signal \( s_1 \) is transmitted and \( H_2 \) the event that \( s_2 \) is transmitted. The received signal differs from the transmitted signal in that there is a random phase term included and because of the noise. If \( s_1 \) is transmitted the received signal then is

\[ r(t) = a_1(t) \cos(\omega_1 t + \beta_1(t) + \Theta_1) + n(t) \]

where \( n(t) \) is a white Gaussian noise process. If \( s_2 \) is transmitted the received signal then is

\[ r(t) = a_2(t) \cos(\omega_2 t + \beta_2(t) + \Theta_2) + n(t), \]

We would like to compute the error probability of the optimal receiver. The optimal receiver processes the received signal by correlating with two signals and then sums the squares. That is the receiver first computes

\[ X_{0,t} \triangleq \int r(t) a_0(t) \cos(\omega_0 t + \beta_0(t)) dt \]

and

\[ X_{1,t} \triangleq \int r(t) a_1(t) \sin(\omega_0 t + \beta_0(t)) dt \]

Then

\[ X_t = X_{0,t}^2 + X_{1,t}^2. \]

The receiver decides signal 2 was transmitted if \( X_2 \geq X_1 \) and otherwise decides \( s_1 \) transmitted. The probability of error given signal \( s_2 \) is transmitted is then

\[ P(\text{error} \mid H_2) = P(X_2 \geq X_1 \mid H_2) \]

To calculate the error probability we need to know the density of \( X_t \). It is easy to see that \( X_{0,c} \) and \( X_{1,c} \) are a Gaussian random variables with mean

\[ E[X_{0,c} \mid H_m, \theta_m] = \frac{1}{2} \rho_{0m} \cos \theta_m \]

\[ E[X_{1,c} \mid H_m, \theta_m] = \frac{1}{2} \rho_{1m} \sin \theta_m \]
and variance
\[ \text{Var}[X_k|H_m, \theta_m] = \frac{1}{4} N_0 E \]
where \( E \) is the energy of the transmitted signal. The density of \( X_k \) given \( H_m \) can then be calculated in a straightforward manner as
\[ \rho_m(x_k) = \frac{1}{2\sigma^2} \exp \left\{ -\frac{(\mu^2 + x_k)}{2\sigma^2} \right\} I_0 \left( \frac{\sqrt{2}\mu x_k}{\sigma^2} \right), \quad x > 0 \]
where
\[ \mu \triangleq \mu^2 + \mu_c^2 \]
\[ \mu_c \triangleq \frac{1}{2} p_{h,m} \cos \theta_k \]
\[ \mu_s \triangleq \frac{1}{2} p_{h,m} \sin \theta_k \]
\[ \sigma^2 \triangleq \frac{1}{4} N_0 E \]

Involved calculation then yields
\[ P\{\text{error} | H_1\} = P\{X_2 \geq X_1 | H_1\} = Q(a, b) = \frac{1}{2} \exp\{ (a^2 + b^2)/2 \} I_0(ab) \]

Bit Error Probability of Nonorthogonal Signals

Figure 25: Performance of Nonorthogonal Signals with Noncoherent Demodulation
Error Probability for \(M\)-Orthogonal Signals: Noncoherent Reception in AWGN

**Preliminaries:** First we derive the density for the sum of the squares of two Gaussian random variables. Let

\[X_i \sim N(\mu_i, \sigma^2_i)\]
\[\sigma_i \sim N(\mu_i, \sigma^2_i)\]

with \(X_i, \sigma_i\) independent. Let \(\mu^2 = \mu_i^2 + \mu_i^2\) and

\[Y = X_i^2 + X_i^2.\]

Then

\[P(Y \leq y) = \int_{x_i^2 + x_i^2 \leq y} \frac{1}{2\sigma^2} \exp\left\{-\frac{1}{\sigma^2} \left[ (x_i - \mu_i)^2 + (x_i - \mu_i)^2 \right]\right\} \, dx_i \, dx_i\]

\[= \int_{x_i^2 + x_i^2 \leq y} \frac{1}{2\sigma^2} \exp\left\{-\frac{1}{\sigma^2} \left[ x_i^2 + x_i^2 - 2(x_i \mu_i + x_i \mu_i) + \mu^2 \right]\right\} \, dx_i \, dx_i\]

Let \(u = r^2\) then \(0 \leq r \leq \sqrt{y}\) is equivalent to \(u \leq y\). Also \(du = 2rdr\).

\[P(Y \leq y) = \int_{u \leq y} \frac{1}{2\sigma^2} \exp\left\{-\frac{u + \mu^2}{2\sigma^2}\right\} \frac{1}{\sqrt{u}} \, du\]

\[f_Y(y) = \frac{1}{2\sigma^2} \exp\left\{-\frac{y + \mu^2}{2\sigma^2}\right\} \frac{1}{\sqrt{y}} \, \frac{1}{\sqrt{2\pi}}\]

A change of variables makes for a cleaner expression: Let \(W = Y/(2\sigma^2)\). Then \(f_W(w) = 2\sigma^2 f_Y(2\sigma^2 w)\)

\[f_W(w) = \exp\left\{-\frac{(w + \Gamma)}{2\sigma^2}\right\} \frac{1}{\sqrt{4\pi w}}\]

where \(\Gamma = \mu^2/(2\sigma^2)\). (If the receiver does this normalization then it must know the power density of the noise.) Now let \(Z = \sqrt{Y}\). Then

\[P(Z \leq z) = P\left\{\sqrt{Y} \leq z\right\} = P\{Y \leq z^2\}\]

\[F_Z(z) = F_Y(z^2)\]

\[f_Z(z) = f_Y(z^2)(2z)\]

\[= \frac{z}{\sigma^2} \exp\left\{-\frac{z^2 + \mu^2}{2\sigma^2}\right\} \frac{1}{\sqrt{2\pi}}\]
\[ \mu = 0 \implies f_1(y) = \frac{1}{2\sigma^2} e^{-y^2/2\sigma^2} \]

\[ f_2(z) = \frac{z}{\sigma^2} e^{-z^2/2\sigma^2} \]

Doing a change of variables \((v = \frac{z}{\sigma}, \; dv = \frac{dz}{\sigma})\) we obtain

\[ P(Z \leq z) = \int_{u \leq z} \frac{u}{\sigma^2} e^{u^2/2\sigma^2} du = \int_{v \leq z} e^{v^2/2\sigma^2} dv \]

\[ = 1 - e^{-z^2/2\sigma^2} \]

\[ P_{c,d} = \int_0^\infty p(z) \left( 1 - e^{-z^2/2\sigma^2} \right) dz \]

\[ = \int_0^\infty p(z) \left( \sum_{l=0}^{M-1} (-1)^l \left( \frac{M-1}{l} \right) e^{-l z^2/2\sigma^2} \right) dz \]

\[ = \sum_{l=0}^{M-1} (-1)^l \left( \frac{M-1}{l} \right) \int_0^\infty p(z) e^{-l z^2/2\sigma^2} dz \]

\[ = \sum_{l=0}^{M-1} (-1)^l \left( \frac{M-1}{l} \right) \int_0^\infty \frac{z}{\sigma^2} e^{-l z^2/2\sigma^2} \left( \frac{\sigma^2}{2\pi} \right) dz \]

Do another change of variables, \((w_i = \sqrt{l+1} z, \; dw_i = \sqrt{l+1} dz)\) we get

\[ \int_0^\infty p(z) e^{-l z^2/2\sigma^2} dz \]

\[ = e^{-l^2/2\sigma^2} \int_0^\infty \frac{w_i}{\sigma^2} e^{-l (w_i^2/2\sigma^2)} \left( \frac{\sigma^2}{2\pi} \right) \left( \frac{\mu_{w_i} \sigma^2}{\sqrt{l+1}} \right) \left( \frac{\sqrt{l+1}}{\sigma} \right) dw_i \]

\[ = e^{-l^2/2\sigma^2} \int_0^\infty \frac{w_i}{\sigma^2} e^{-l (w_i^2/2\sigma^2)} \left( \frac{\sigma^2}{2\pi} \right) \left( \frac{\mu_{w_i} \sigma^2}{\sqrt{l+1}} \right) \left( \frac{\sqrt{l+1}}{\sigma} \right) dw_i \]

Let \(Z = \sqrt{X_{j_k}^2 + X_{\hat{j}_k}^2}\). Then we need to determine the probability that \(Z_0\) which corresponds to nonzero mean random variables is less than \(Z_1, Z_2, \ldots, Z_{M-1}\) which correspond to zero mean random variables.

\[ P_{c,d} = P(Z \leq Z_i, j \neq i | s_i \text{ transmitted}) \]

\[ = E[P(Z \leq Z_i, j \neq i | Z_i, s_i \text{ trans}, Z_j)] \]

\[ = \int p(z) [P(Z \leq z)]^{M-1} dz \]

\[ = e^{-l^2/2\sigma^2} \int_0^\infty \frac{w_i}{\sigma^2} \left( \frac{\sigma^2}{2\pi} \right) \left( \frac{\mu_{w_i} \sigma^2}{\sqrt{l+1}} \right) \left( \frac{\sqrt{l+1}}{\sigma} \right) dw_i \]

Let \(\hat{\mu} = \frac{\mu}{\sqrt{1+\lambda}}\). Then

\[ \int_0^\infty p(z) e^{-l z^2/2\sigma^2} dz = e^{-l^2/2\sigma^2} \int_0^\infty \frac{w_i}{\sigma^2} e^{-l (w_i^2/2\sigma^2)} \left( \frac{\sigma^2}{2\pi} \right) \left( \frac{\mu_{w_i} \sigma^2}{\sqrt{l+1}} \right) \left( \frac{\sqrt{l+1}}{\sigma} \right) dw_i \]

\[ = \exp \left( \frac{\hat{\mu}^2}{2\sigma^2} \right) \exp \left( \frac{\lambda^2}{2(l+1)^2} \right) \frac{1}{l+1}. \]

Substituting this into the expression for the probability of correct, we obtain

\[ P_{c,d} = \sum_{l=0}^{M-1} (-1)^l \left( \frac{M-1}{l} \right) \exp \left( \frac{\lambda^2}{2(l+1)^2} \right) \frac{1}{l+1}. \]
where

\[ \frac{\mu^2}{2\sigma^2} = \frac{E^2}{2N_0} = \frac{E}{N_0}. \]

Thus

\[ P_{c,i} = 1 + \sum_{l=1}^{M-1} \left( -1 \right)^l \binom{M}{l} \exp \left\{ -\frac{lE}{(l+1)N_0} \right\} \]

\[ P_{e,i} = 1 - P_{c,i} = \sum_{l=1}^{M-1} \left( -1 \right)^{l+1} \binom{M}{l} \exp \left\{ -\frac{lE}{l+1N_0} \right\} \]

\[ P_e = \frac{M-1}{M} \sum_{i=1}^{M} P_{e,i} = P_{c,i} \]

\( M \) signals \( \Rightarrow \) \( \log_2 M \) bits

The limiting behavior of the error probability for \( M \)-ary orthogonal signals with noncoherent demodulation is the same as the limiting performance of coherent demodulation. If \( M = 2^k \)

\[ P_{e,b} = \frac{1}{M} \sum_{i=1}^{M} P_{e,i} = \frac{E_b}{\log_2 M(R)} \]

It can be shown that the asymptotic behavior of \( M \) orthogonal signals on an additive Gaussian noise channel with noncoherent reception is the same as with coherent reception. That is for \( E_b/N_0 < \ln 2 \) the error probability is 1 while for \( E_b/N_0 > \ln 2 \) the error probability is 0.

![Error Probability for \( M \) orthogonal signals](image)

Figure 26: Bit error probability of \( M \)-ary orthogonal modulation in an additive white Gaussian noise channel with noncoherent demodulation

**Error Estimates for Repetition Codes with Noncoherent Reception**

Consider transmitting one of two codewords of length \( L \) using binary frequency shift keying (orthogonal) and noncoherent reception. The optimum receiver would compute the log-likelihood ratio and compare to zero (for equally likely codewords) to determine which of the two codewords was transmitted. Assume that the first codeword is the all zero vector \((0,0,...,0)\) of length \( L \) and the other codeword is the all one vector \((1,1,...,1)\) of length \( L \). If the two codewords agreed in some positions then we would not need to process the received signal over the interval of time where they agreed since no information can be gained about which signal was transmitted from the received signal in that interval.

The transmitted signal would be

\[ s_0(t) = \sum_{l=1}^{L} \sqrt{2P} \cos(\omega_0 t + \theta_0) p_l(t - lT) \]
or

\[ s_1(t) = \sum_{l=0}^{L-1} \sqrt{2P} \cos(\omega t + \theta_{l,t}) p_r(t - lT) \]

where \( \theta_{l,t} \) are independent identically distributed random variables with uniform distribution unknown to the receiver. The received signal is

\[ r(t) = \begin{cases} 
  s_0(t) + n(t) & H_0 \text{ true} \\
  s_1(t) + n(t) & H_1 \text{ true} 
\end{cases} \]

The receiver processes the signals using noncoherent matched filters; that is the received signal is multiplied by \( j\omega \) then passed through a filter with impulse response \( p_r(t) \), sampled and then the magnitude is formed. Denote by \( Y_{0,l} \) the output of the noncoherent matched filter

\[ Y_{0,l} = \left| \int_{-\infty}^{\infty} r(s) p_r(s - lT) \exp\{j\omega(s - lT)\} ds \right|^2 \]

and

\[ Y_{1,l} = \left| \int_{-\infty}^{\infty} r(s) p_r(s - lT) \exp\{j\omega(s - lT)\} ds \right|^2 \]

The optimum receiver is thus

\[ \sum_{l=1}^{L} \log[p(y_{0,l} | H_0)] - \sum_{l=1}^{L} \log[p(y_{1,l} | H_1)] \]

which is quite complex in implementation. It would be useful to have suboptimum receivers which are easier to implement but have nearly optimum performance. Before we suggest some suboptimal receivers is would be useful to get estimates of the performance of the optimum receiver. The error probability (given \( H_0 \) say) is easy to write down but hard to evaluate except for \( L \) small. It is

\[ P_e = \int_y I[R_1] p(y | H_0) dy \]

where \( I[R_1] \) is the region where the log-likelihood ratio is positive. This is a 2L dimensional integral. The Chernoff bound to the error probability can be expressed as

\[ P_e \leq D^L \]

where

\[ D = \int_y \sqrt{p(y | H_0)p(y | H_1)} dy, \]

The statistics of \( Y_{0,l}, Y_{1,l} \) were calculated in the previous section. Because of orthogonality of the received signals and that the noise is white the joint statistics of \( Y = (Y_{0,0}, ..., Y_{0,L}, Y_{1,1}, ..., Y_{1,L}) \) factor (conditioned on either of the two hypotheses) as

\[ p(y_{0,1}, ..., y_{0,L}, y_{1,1}, ..., y_{1,L} | H_k) = \prod_{l=0}^{L} p(y_{0,l} | H_k)p(y_{1,l} | H_k) \quad k = 0, 1, \]

The log-likelihood ratio is then

\[ \Lambda = \log \frac{p(y | H_1)}{p(y | H_0)} = \log \prod_{l=1}^{L} \frac{p(y_{0,l} | H_1)p(y_{1,l} | H_1)}{p(y_{0,l} | H_0)p(y_{1,l} | H_0)} \]

\[ = \sum_{l=1}^{L} \log \frac{p(y_{0,l} | H_1)p(y_{1,l} | H_1)}{p(y_{0,l} | H_0)p(y_{1,l} | H_0)} \]

Substituting in the appropriate density functions yields

\[ \Lambda = \sum_{l=1}^{L} \log \left[ \int_y \frac{p(y_{0,l})}{p(y_{1,l})} \right] = \log \left[ \int_y \frac{p(y_{0,l})}{p(y_{1,l})} \right] \]

For the additive white Gaussian channel

\[ D = \left[ \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left( -\frac{1}{2} \frac{y^2}{\sigma^2} \right) \sqrt{h_0(\frac{\mu y}{\sigma^2})} dy \right]^2 \]

\[ = \left[ \int_0^\infty \exp\left( -\frac{(w - \Gamma/2)^2}{\Gamma \sigma^2} \right) \sqrt{h_0(\sqrt{4\Gamma}w)} dw \right]^2 \]

and \( \Gamma = E/N_0 \). This is much more computationally attractive than the exact expression. An asymptotic form for low signal-to-noise ratios is not to difficult to compute.

\[ D \approx 1 - \left( \frac{\Gamma}{2} \right)^2 \quad \Gamma \text{ small}, \]

For hard decisions

\[ D = 2\sqrt{p(1-p)}, \quad p = \frac{1}{2} e^{-\frac{\Gamma}{2}} \]

which for low signal-to-noise ratios becomes

\[ D \approx 1 - \left( \frac{\Gamma}{2\sqrt{2}} \right)^2 \quad \Gamma \text{ small}, \]

Thus for low signal-to-noise ratios hard decisions cost \( \sqrt{2}=1.5 \text{dB} \).
Now let us consider some suboptimal receivers. First note that \( \mu / \sigma^2 = 2/\text{N}_0 \). So the argument to the Bessel function will be small if the noise power is large (low signal-to-noise ratio) and will be large if the noise power is small (high signal-to-noise ratio). Also note that (see Abramowitz and Stegun Handbook of Mathematical Functions)

\[
I_0(x) \approx 1 + \frac{x^2}{4}, \quad x \text{ small}
\]

\[
\log I_0(x) \approx \frac{x^2}{4}, \quad x \text{ small}
\]

\[
I_0(x) \approx e^{x} / \sqrt{2\pi x}, \quad x \text{ large}
\]

Using the approximation for \( \log I_0 \) in the optimum decision rule we get

\[
\sum_{l=1}^{L} \left( \frac{w_0}{4\sigma^2} \right) \frac{y_{i,l}}{\sigma_l} \left( \frac{w_0}{4\sigma^2} \right) \frac{y_{i,l}}{\sigma_l}
\]

\[
\sum_{l=1}^{L} y_{i,l} \frac{y_{i,l}}{\sigma_l} \left( \frac{w_0}{4\sigma^2} \right) \frac{y_{i,l}}{\sigma_l}
\]

So for small signal-to-noise ratios the optimum receiver is the square-law combining receiver.

Now consider when the argument to the Bessel function is large. The optimum decision rule

the sum of the squares of \( 2L \) random variables. Let

\[
W_0 = \frac{1}{2\sigma^2} \sum_{l=1}^{L} X_{i,l}^2 + X_{i,l}^2
\]

and similarly for \( W_1 \). Then the density of \( W \) is given by

\[
f_{w_0}(w_0|H_0) = \left( \frac{w_0}{\Gamma} \right)^{L/2} \exp\{- w_0 + \Gamma \} I_L \left( \sqrt{4w_0\Gamma} \right), \quad w \geq 0
\]

\[
f_{w_0}(w_0|H_1) = \left( \frac{w_0}{\Gamma} \right)^{L/2} \exp\{- w_0 \}, \quad w \geq 0
\]

and similarly for \( W_1 \).

\[
P_e = 1 - \Pr\{W_0 > W_1|H_0\}
\]

\[
= 1 - \int_{0}^{\infty} f_{w_0}(w_0|H_0) P\{W_1 < w_0|H_0\} \, dw_0
\]

\[
= 1 - \int_{0}^{w_0} f_{w_0}(w_0|H_0) \left[ 1 - \sum_{m=0}^{L-1} \frac{w_0^m}{m!} e^{-w_0} \right] \, dw_0
\]

\[
P_e = \frac{1}{2} \exp\{- \frac{\Gamma^2}{2} \} \sum_{m=0}^{L-1} \frac{(-\Gamma/2)^m}{m!} \left( \frac{L}{L-1} \right)^{m} \sum_{j=0}^{L-1} \frac{(-j)^{j+m}}{j!}
\]

Then becomes

\[
\sum_{l=1}^{L} \left( \frac{\mu \sqrt{\sum_{i=1}^{N} y_{i,l}^2}}{2\sigma^2} \right) - \log \sqrt{2\pi \mu \sqrt{\sum_{i=1}^{N} y_{i,l}^2} / \sigma^2} = \left( \frac{\mu \sqrt{\sum_{i=1}^{N} y_{i,l}^2}}{2\sigma^2} \right) + \log \sqrt{2\pi \mu \sqrt{\sum_{i=1}^{N} y_{i,l}^2} / \sigma^2} \leq \frac{H}{\text{N}_0} \leq 0
\]

Let \( w_{ij} = y_{ij} / (2\sigma^2) \) then the decision rule is

\[
\sum_{l=1}^{L} \left( \sqrt{4\Gamma w_{ij,l}} \right) - \log \sqrt{\frac{2\pi \sqrt{4\Gamma w_{ij,l}}}{2\sigma^2} / \sigma^2} = \left( \sqrt{4\Gamma w_{ij,l}} \right) + \log \sqrt{2\pi \sqrt{4\Gamma w_{ij,l}} / \sigma^2} \leq \frac{H_j}{\text{N}_0} \leq 0
\]

Note that the average value of \( W \) given signal present is \( \Gamma + 1 \) while the average value of \( W \) given no signal is 1. For very large \( \Gamma \) the terms \( \sqrt{4\Gamma w} \) dominates the terms of the form \( \log \sqrt{2\pi \sqrt{4\Gamma w}} \) and thus the optimum decision rule is

\[
\sum_{l=1}^{L} \left( \sqrt{4\Gamma w_{ij,l}} \right) \geq \frac{H_j}{\text{N}_0} \sum_{l=1}^{L} \left( \sqrt{w_{ij,l}} \right)
\]

Thus the decision rule for very high signal-to-noise ratio is to add the square-roots of the noncoherent matched filter outputs.

It is of interest to analyze the performance of these suboptimal receivers. The receiver for very low signal-to-noise ratios is (relatively) easy to analyze. First let us normalize the density for

For \( L = 1 \) the above becomes

\[
P_e = \frac{1}{2} e^{-\frac{1}{2} \frac{1}{\Gamma^2}}
\]

where \( \Gamma = E/\text{N}_0 \). The Chernoff bound can also be calculated for square-law combining.
Primer on sums of squares of Gaussian random variables

First we derive the density for the sum of the squares of two Gaussian random variables. Let

$$X_c \sim N(\mu_c, \sigma_c^2)$$
$$X_s \sim N(\mu_s, \sigma_s^2)$$

with $X_c, X_s$ independent. Let $\mu^2 = \mu_c^2 + \mu_s^2$ and

$$Y = X_c^2 + X_s^2.$$

Then

$$P(Y \leq y) = \int_{x_c^2 + x_s^2 \leq y} \frac{1}{2\pi \sigma_c^2} \exp \left\{ -\frac{1}{2\sigma_c^2} [(x_c - \mu_c)^2 + (x_s - \mu_s)^2] \right\} \, dx_c \, dx_s$$

$$= \int_{x_c^2 + x_s^2 \leq y} \frac{1}{2\pi \sigma_c^2} \exp \left\{ -\frac{1}{2\sigma_c^2} [x_c^2 + x_s^2 - 2(x_c \mu_c + x_s \mu_s) + \mu^2] \right\} \, dx_c \, dx_s$$

$$= \int_{x_c^2 + x_s^2 \leq y} \frac{1}{2\pi \sigma_c^2} \exp \left\{ -\frac{1}{2\sigma_c^2} [x_c^2 + x_s^2 - 2\mu \sqrt{x_c^2 + x_s^2}] \right\} \, dx_c \, dx_s$$

where $\beta = \sqrt{\mu_c^2 + \mu_s^2} \sqrt{x_c^2 + x_s^2}$

$$x_c \mu_c + x_s \mu_s = \sqrt{\mu_c^2 + \mu_s^2} \sqrt{x_c^2 + x_s^2} \cos(\phi + \gamma)$$

$$\phi = \tan^{-1} \left( \frac{x_c}{x_s} \right)$$

$$\gamma = \tan^{-1} \left( \frac{-\mu_s}{\mu_c} \right)$$

$$P(Y \leq y) = \int_{\sqrt{x_c^2} \leq \sqrt{y}} \frac{r}{\sigma_c} \exp \left\{ -\frac{r^2 + \mu^2}{2\sigma_c^2} \right\} I_0 \left( \frac{\mu^2}{\sigma_c^2} \right) \, dr$$

Let $u = r^2$ then $0 \leq u \leq \sqrt{y}$ is equivalent to $u \leq y$. Also $du = 2u \, dr$.

$$P(Y \leq y) = \int_{u \leq y} \frac{1}{2\sigma_c^2} \exp \left\{ -\frac{u + \mu^2}{2\sigma_c^2} \right\} I_0 \left( \frac{\mu \sqrt{u}}{\sigma_c^2} \right) \, du$$

$$f_Y(y) = \frac{1}{2\sigma_c^2} \exp \left\{ -\frac{y + \mu^2}{2\sigma_c^2} \right\} I_0 \left( \frac{\mu \sqrt{y}}{\sigma_c^2} \right)$$

A change of variables makes for a cleaner expression: Let $W = Y/(2\sigma_c^2)$. Then

$$f_W(w) = 2\sigma_c^2 f_Y(2\sigma_c^2 \cdot w)$$

$$f_W(w) = \exp \left\{ -(w + \Gamma) \right\} I_0 \left( \sqrt{4\Gamma w} \right)$$

where $\Gamma = \mu^2/(2\sigma_c^2)$. (If the receiver does this normalization then it must know the power density of the noise). Now let $Z = \sqrt{W}$. Then

$$P(Z \leq z) = P \left\{ \sqrt{W} \leq z \right\} = P \{ Y \leq z^2 \}$$

$$f_Z(z) = \frac{f_Y(z^2)}{2z}$$

$$f_Z(z) = \frac{\frac{z}{\sqrt{\Gamma}} \exp \left\{ -\frac{z^2 + \mu^2}{2\sigma_c^2} \right\} I_0 \left( \frac{\mu \sqrt{z}}{\sigma_c^2} \right)}{2z}$$

$$\mu = 0 \Rightarrow f_Y(y) = \frac{1}{2\sigma_c^2} e^{-y/2\sigma_c^2}$$

$$f_Z(z) = \frac{\frac{z}{\sqrt{\Gamma}} e^{-z^2/(2\sigma_c^2)}}{2z}$$

Using the fact that a density must integrate to one we can derive an useful integral.

$$\int_0^\infty \frac{r}{\sigma_c^2} \exp \left\{ -\frac{r^2}{2\sigma_c^2} \right\} \exp \left\{ -\frac{\mu^2}{2\sigma_c^2} \right\} I_0 \left( \frac{\mu \sqrt{r}}{\sigma_c^2} \right) \, dr = \frac{1}{1 + 2\alpha^2} \exp \left\{ -\frac{\sigma_c^2 \beta^2}{1 + 2\alpha^2} \right\}$$
Generalization:

\[ X_{e,i} \sim N(\mu_{e,i}, \sigma^2) \quad i = 1, 2, ..., L \]
\[ X_{o,i} \sim N(\mu_{o,i}, \sigma^2) \quad i = 1, 2, ..., L \]

with \( X_{e,i}, X_{o,i} \) independent. Let 
\[ \Lambda = \sum_{i=1}^{L} \theta_{e,i}^2 + \theta_{o,i}^2 \text{ and } \]
\[ Y = \sum_{i=1}^{L} X_{e,i}^2 + X_{o,i}^2. \]

Then
\[ f_Y(y) = \frac{1}{2\sigma^2} \exp\left(-\left(\frac{y + \Lambda}{2\sigma^2}\right)\right) \left(\frac{y}{\Lambda}\right)^{(L-1)/2} I_L(\sqrt{\frac{\Lambda}{\sigma^2}}) \]
and
\[ F_Y(y) = 1 - Q_L\left(\sqrt{\frac{\Lambda}{\sigma^2}}, \sqrt{\frac{y}{\sigma}}\right) \]
where
\[ Q_L(a, b) = Q(a, b) + \exp\left(\frac{a^2 + b^2}{2}\right) \sum_{k=1}^{L} \frac{b^k}{k!} I_k(ab) \]

and
\[ Q(a, b) = \exp\left(-\frac{a^2 + b^2}{2}\right) \sum_{k=1}^{L} \frac{b^k}{k!} I_k(ab) \]

Consider two random variables \( Z_1 \) and \( Z_2 \) where \( Z_1 \) has distribution given above with \( \Lambda_1 = 0 \)

For \( \Lambda = 0 \)
\[ f_Y(y) = \frac{1}{2\sigma^2} \exp\left(-\left(\frac{y}{2\sigma^2}\right)\right) \left(\frac{y}{\sigma^2}\right)^{(L-1)/2} \cdot \frac{1}{(L-1)!} \]
\[ F_Y(y) = 1 - \exp\left(-\left(\frac{y}{2\sigma^2}\right)\right) \sum_{k=0}^{L-1} \frac{1}{k!} \left(\frac{y}{2\sigma^2}\right)^k \]

Let \( Z = \sqrt{Y} \) then
\[ f_Z(z) = \frac{z^{L-1/2}}{\sigma^2} \exp\left(-\left(\frac{z^2 + \Lambda}{2\sigma^2}\right)\right) I_L\left(\sqrt{\frac{\Lambda}{\sigma^2}}\right) \]
\[ F_Z(z) = 1 - Q_L\left(\sqrt{\frac{\Lambda}{\sigma^2}}, \sqrt{\frac{z}{\sigma}}\right) \]

For \( \Lambda = 0 \) we obtain
\[ f_Z(z) = \frac{z^{L-1/2}}{2^{L-1} \sigma^2 (L-1)!} \exp\left(-\frac{z^2}{2 \sigma^2}\right) \]
\[ F_Z(z) = 1 - \exp\left(-\frac{z^2}{2 \sigma^2}\right) \sum_{k=0}^{L-1} \frac{1}{k!} \left(\frac{z^2}{2 \sigma^2}\right)^k \]

and with different variances \( \sigma_1 \) and \( \sigma_2 \). Assume that they are independent. We wish to determine the probability that \( Z_1 > Z_2 \).

\[ P\{Z_1 < Z_2\} = \int_{z_1 = 0}^{\infty} \int_{z_2 = 0}^{z_1} f_{Z_1}(z_1) f_{Z_2}(z_2) dz_1 dz_2 \]
\[ = \int_{z_1 = 0}^{\infty} f_{Z_1}(z_1) \left[ \int_{z_2 = 0}^{z_1} f_{Z_2}(z_2) dz_2\right] dz_1 \]
\[ = \int_{z_1 = 0}^{\infty} f_{Z_1}(z_1) \int_{z_2 = 0}^{z_1} \frac{1}{\sqrt{2\pi} \sigma_1} \exp\left(-\frac{z_2^2}{2 \sigma_1^2}\right) \frac{1}{\sqrt{2\pi} \sigma_2} \exp\left(-\frac{z_2^2}{2 \sigma_2^2}\right) \]
\[ = \int_{z_1 = 0}^{\infty} \frac{1}{\sqrt{2\pi} \sigma_1} \exp\left(-\frac{z_1^2}{2 \sigma_1^2}\right) \int_{z_2 = 0}^{z_1} \frac{1}{\sqrt{2\pi} \sigma_2} \exp\left(-\frac{z_2^2}{2 \sigma_2^2}\right) \]
\[ = \int_{z_1 = 0}^{\infty} \frac{1}{\sqrt{2\pi} \sigma_1} \exp\left(-\frac{z_1^2}{2 \sigma_1^2}\right) \int_{z_2 = 0}^{\infty} \frac{1}{\sqrt{2\pi} \sigma_2} \exp\left(-\frac{z_2^2}{2 \sigma_2^2}\right) \]
\[ = \frac{1}{\sqrt{2\pi} \sigma_1} \exp\left(-\frac{z_1^2}{2 \sigma_1^2}\right) \int_{z_2 = 0}^{\infty} \frac{1}{\sqrt{2\pi} \sigma_2} \exp\left(-\frac{z_2^2}{2 \sigma_2^2}\right) \]
\[ = \frac{1}{\sqrt{2\pi} \sigma_1} \exp\left(-\frac{z_1^2}{2 \sigma_1^2}\right) \exp\left(-\frac{z_2^2}{2 \sigma_2^2}\right) \]
\[ = \frac{1}{\sqrt{2\pi} \sigma_1} \exp\left(-\frac{z_1^2}{2 \sigma_1^2}\right) \frac{1}{\sqrt{2\pi} \sigma_2} \exp\left(-\frac{z_2^2}{2 \sigma_2^2}\right) \]
\[ = \frac{1}{\sqrt{2\pi} \sigma_1} \exp\left(-\frac{z_1^2}{2 \sigma_1^2}\right) \frac{1}{\sqrt{2\pi} \sigma_2} \exp\left(-\frac{z_2^2}{2 \sigma_2^2}\right) \]
\[ = \frac{1}{\sqrt{2\pi} \sigma_1} \frac{1}{\sqrt{2\pi} \sigma_2} \exp\left(-\frac{z_1^2}{2 \sigma_1^2}\right) \frac{1}{\sqrt{2\pi} \sigma_2} \exp\left(-\frac{z_2^2}{2 \sigma_2^2}\right) \]

where
\[ \alpha^2 = \frac{1}{2\sigma_1^2} + \frac{1}{2\sigma_2^2} \]
\[ \gamma = \sqrt{\lambda/\sigma_1^2} \]

The integral may be evaluated as (see Lindsey, Watson)
\[ \int_{z_1 = 0}^{\infty} e^{-z_1^2/2} \frac{1}{\sqrt{2\pi} \sigma_1} I_L\left(\frac{z_1^2}{2 \sigma_1^2}\right) dz_1 = \frac{\Gamma\left(L+\frac{1}{2}\right)}{2^{L+\frac{1}{2}} \pi^{L/2} \alpha^2} e^{2\lambda/4\alpha^2} \sum_{k=0}^{L} \left(\frac{L+k}{L-1}\right) \frac{1}{k!} \left(\frac{\lambda}{2 \alpha^2}\right)^k \]

Thus
\[ P\{Z_1 > Z_2\} = \sum_{k=0}^{L} \frac{1}{\sigma_2^2 \alpha^2} \frac{1}{\sqrt{2\pi} \sigma_2} \exp\left(-\frac{z_2^2}{2 \sigma_2^2}\right) \frac{1}{\sqrt{2\pi} \sigma_1} \frac{1}{\sqrt{2\pi} \sigma_2} \exp\left(-\frac{z_2^2}{2 \sigma_2^2}\right) \]
\[ = \sum_{k=0}^{L} \frac{1}{\sigma_2^2 \alpha^2} \frac{1}{\sqrt{2\pi} \sigma_2} \exp\left(-\frac{z_2^2}{2 \sigma_2^2}\right) \frac{1}{\sqrt{2\pi} \sigma_1} \frac{1}{\sqrt{2\pi} \sigma_2} \exp\left(-\frac{z_2^2}{2 \sigma_2^2}\right) \]
\[ = \sum_{k=0}^{L} \frac{1}{\sigma_2^2 \alpha^2} \frac{1}{\sqrt{2\pi} \sigma_2} \exp\left(-\frac{z_2^2}{2 \sigma_2^2}\right) \frac{1}{\sqrt{2\pi} \sigma_1} \frac{1}{\sqrt{2\pi} \sigma_2} \exp\left(-\frac{z_2^2}{2 \sigma_2^2}\right) \]
\[ \sum_{k=0}^{l} \left( \frac{(l+L-1)}{l-k} \right) \left( \frac{\gamma^2}{(4\alpha^2)} \right)^{k} \]

\[ \frac{1}{(2\sigma_{1}^{2}+\sigma_{2}^{2})} \sum_{k=0}^{l} \exp \left( \frac{-\Lambda}{2(\sigma_{1}^{2}+\sigma_{2}^{2})} \right) \left( \frac{\gamma^2}{(4\alpha^2)} \right)^{k} \]

\[ \frac{\gamma^2}{(4\alpha^2)} = \frac{(\Lambda/\sigma_{1}^{2})2\sigma_{1}^{2}/\sigma_{1}^{2} + \sigma_{2}^{2}}{\sigma_{1}^{2} + \sigma_{2}^{2}} \]

\[ \frac{\gamma^2}{(4\alpha^2)} - \frac{\Lambda/\sigma_{1}^{2}}{\sigma_{1}^{2} + \sigma_{2}^{2}} = \frac{\Lambda/\sigma_{1}^{2}}{\sigma_{1}^{2} + \sigma_{2}^{2}} \left( \frac{\sigma_{1}^{2}}{\sigma_{1}^{2} + \sigma_{2}^{2}} - 1 \right) \]

Performance as a function of L

Add curves for various L

---

**Frequency Shift Keying (FSK)**

Frequency shift keying communicates information by transmitting different frequencies. It can be demodulated noncoherently (by measuring the received energy at the different frequencies). It performance is worse than coherently demodulated signals but may be simpler.

\[ b(t) \]

\[ VCO \]

\[ s(t) \]

\[ \pm \]

\[ r(t) \]

Figure 27: FSK Modulator

\[ b(t) = \sum_{n} b_{n} p_{F}(t - nT), \quad b_{n} \in \{+1, -1\} \]
\[ s(t) = \sqrt{2P} \sum_{l=-\infty}^{\infty} \cos(2\pi(f_c + b(t)\Delta f) + \theta) p_{\gamma}(t - IT) \]

where \( \Delta f \) is half the difference between the two transmitted frequencies and \( \theta \) is an unknown (to the receiver) phase. We let \( f_0 = f - \Delta f \) and \( f_1 = f + \Delta f \). When \( b_i = +1 \) then a signal at frequency \( f_1 \) is transmitted. When \( b_i = -1 \) then a signal at frequency \( f_0 \) is transmitted. The two frequencies \( f_0 \) and \( f_1 \) are separated far enough to make the two signals orthogonal. (Minimum shift keying has the minimum separation in order to make the signals orthogonal).

The receiver decides signal \(-1\) was transmitted if \( |Y_1| > |Y_1| \) and otherwise decides signal 1. The random variables at the output of the low pass filters are

\[
\begin{align*}
X_{c,1}(IT) &= \sqrt{E}\delta(b_i, +1) \cos(\theta) + \eta_{c,1} \\
X_{s,1}(IT) &= \sqrt{E}\delta(b_i, +1) \sin(\theta) + \eta_{s,1} \\
X_{i,1}(IT) &= \sqrt{E}\delta(b_i, -1) \cos(\theta) + \eta_{i,1} \\
X_{c,1}(IT) &= \sqrt{E}\delta(b_i, -1) \sin(\theta) + \eta_{c,1}
\end{align*}
\]

where \( \delta(a,b) = 1 \) if \( a = b \) and is zero otherwise. In the absence of noise (\( \eta_{a,i} = 0 \)) it is easy to see that when \( b_i = +1 \) that \( Y_1 = \sqrt{E} \) and \( Y_1 = 0 \). The error probability of binary FSK is

\[ P_{e,b} = \frac{1}{2} e^{-E_0/2N_0} \]

Figure 29: Output Densities For Noncoherent Receivers.
Figure 30: Density for $Y_1 - Y_{-1}$ given +1 Transmitted.

Figure 31: Error Probability of FSK with Noncoherent Detection.

Differential Phase Shift Keying (DPSK)

Differential Encoder is such that

$$b_l = 1 \Rightarrow a_l = a_{l-1},$$

$$b_l = -1 \Rightarrow a_l = -a_{l-1}.$$  

For example

$$l \quad \ldots \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad \ldots$$

$$b_l \quad -1 \quad 1 \quad 1 \quad -1 \quad 1 \quad 1$$

$$a_l \quad 1 \quad -1 \quad 1 \quad 1 \quad -1 \quad -1$$

$$s(t) = \sqrt{2P} a(t) \cos(2\pi f_c t + \theta),$$

where

$$b(t) = \sum_{l=-\infty}^{\infty} b_l p_T (t - lT), \quad b_l \in \{+1, -1\},$$

$$a(t) = \sum_{l=-\infty}^{\infty} a_l p_T (t - lT), \quad a_l \in \{+1, -1\},$$

$$n(t)$$ is the noise.
\[ X_i(iT) = \sqrt{E_d} \cos \theta + \eta_{ci} \]

The random variables \( \eta_{ci} \) and \( \eta_{si} \) are independent identically distributed Gaussian random variables with mean 0 and variance \( N_0/2 \). Thus

\[ Z_i = X_i(iT)X_i((i-1)T) + X_i(iT)X_i((i-1)T) \]

\[ Z_i = \text{Re}[W(iT)W^*(i-1)T)] \]

where \( W(iT) = X_i(iT) - jX_i(iT) \). The error probability for DPSK is

\[ P_{e,b} = \frac{1}{2} e^{-\frac{E_b}{N_0}}. \]

Thus differential phase shift keying is 3dB better than FSK with noncoherent detection. However, errors tend to occur in pairs.

To derive the above expression for DPSK consider the low pass filter with impulse response \( h(t) = p_T(t) \). The output of the lowpass filters can be expressed as

\[ X_c(t) = \int_{-\infty}^{\infty} 2 \cos \omega_c \tau h(t - \tau) r(\tau) d\tau \]

\[ X_s(t) = \int_{-\infty}^{\infty} 2 \cos \omega_s \tau p_T(t - \tau) r(\tau) d\tau \]

\begin{align*}
X_c(iT) &= \int_{(i-1)T}^{iT} 2 \cos \omega_c \tau \left[ \sum_{n=-\infty}^{\infty} \sqrt{2P_d} \cos(\omega_c \tau + \theta) p_T(\tau - iT) + n(\tau) \right] d\tau \\
X_s(iT) &= \int_{(i-1)T}^{iT} \sqrt{2P_d} \cos \omega_s \tau \cos(\omega_s \tau + \theta) d\tau + \eta_{ci} \\
n_{ci} \text{ is Gaussian random variable, mean 0 variance } N_0/2. \text{ Assuming } \omega_s T = 2\pi n \\
X_c(iT) &= \sqrt{2P_d} \cos \theta + \eta_{ci} \\
\end{align*}

Similarly

\[ X_s(iT) = \sqrt{2P_d} \sin \theta + n_{si} \]
Thus
\[ Z_t = X_t(iT)X_e((i - 1)T) + X_e(iT)X_t((i - 1)T) \]

Note that if we write \( W(iT) = X_e(iT) + jX_e(iT) \) that \( Z_t = \text{Re}[W(iT)W^*(i - 1)T] \). It is clear that this represents the phase difference between two consecutive symbols.

Let
\[
\begin{align*}
U_1 &= \frac{X_t(iT) + X_e((i - 1)T)}{2} \\
U_2 &= \frac{X_t(iT) + X_e((i - 1)T)}{2} \\
U_3 &= \frac{X_t(iT) - X_e((i - 1)T)}{2} \\
U_4 &= \frac{X_t(iT) - X_e((i - 1)T)}{2}
\end{align*}
\]

\[ Z_t = U_1^2 + U_2^2 = (U_3^2 + U_4^2) \]

Assume \( b_{t-1} = +1 \) so that \( a_{t-1} = a_{t-2} \) then
\[
\begin{align*}
P_{e0} &= P\{Z < 0 | a_{t-1} = a_{t-2} \} \\
&= P\{U_1^2 + U_2^2 \leq U_3^2 + U_4^2 \} \\
U_1 &\sim N(\mu_1, \sigma^2)
\end{align*}
\]

\[ E[U_1U_2] = \frac{E[X_t(iT)X_e((i - 1)T)]}{2} + \frac{E[X_e(iT)X_t((i - 1)T)]}{2} \]

\[ = \frac{1}{4}E[X_t(iT)X_e(iT) + X_e(iT)X_t((i - 1)T) + X_t((i - 1)T)X_e(iT) + X_e((i - 1)T)X_t((i - 1)T)] \]

\[ U_2 \sim N(\mu_2, \sigma^2) \]

\[ \mu_1 = \frac{1}{2}\sqrt{2p}\left(a_{t-1}T \cos \theta + a_{t-2}T \cos \theta \right) \]

\[ = \frac{1}{2}\sqrt{2p}\left(a_{t-1}T \cos \theta \right) \]

\[ \mu_2 = \frac{1}{2}\sqrt{2p}\left(a_{t-1}T \cos \theta \right) \]

\[ \sigma^2 = \frac{1}{4}[N_0T + N_0T] \]

\[ = \frac{1}{2}N_0T \]

\[ U_1 \sim N(\mu_1, \sigma^2) \]

\[ U_2 \sim N(\mu_2, \sigma^2) \]

\[ \mu_3 = 0, \quad \mu_4 = 0, \]

\[ E[U_1U_2] = E\left[\frac{X_t(iT) + X_e((i - 1)T)}{2}\right] \cdot E\left[\frac{X_e(iT) + X_t((i - 1)T)}{2}\right] \]

\[ = \frac{1}{4}E[\{X_t(iT)X_e(iT) + X_e(iT)X_t((i - 1)T) + X_t((i - 1)T)X_e(iT) + X_e((i - 1)T)X_t((i - 1)T)] \]

\[ P[U_1^2 + U_2^2 \leq U_3^2 + U_4^2] = \frac{1}{2} e^{-\beta|\theta|} \]

Thus differential phase shift keying is 3dB better than FSK with noncoherent detection. However, errors tend to occur in pairs.