Lecture Notes 5: Noncoherent Receivers

Goals

- Derive optimum receiver for arbitary signals in Gaussian noise with a random phase.
- Deterine performance of two signals in white Gaussian noise.
- Deterine performance of M-orthogonal signals in white Gaussian noise.

System Model

Assume additive stationary Gaussian noise and that $s_k(t)$ for $0 \le k \le M-1$ has the form

$$s_k(t) = a_k(t)\cos(\omega_c t + \beta_k(t)), \quad k = 0,1,2,\ldots \le M-1$$

with

$$\int s_k^2(t)dt = \frac{1}{2}\int a_k^2(t)dt = E$$

where $a_k(t)$ and $\beta_k(t)$ are lowpass waveforms with respect to ω_c. When $s_k(t)$ is transmitted the received waveform has the form

$$r(t) = a_k(t)\cos(\omega_c t + \beta_k(t) + \theta_k) + n(t)$$

where θ_k is a random phase. If $\theta_k = 0$ with probability 1 then we have the usual coherent reception situation already discussed. We will for this section assume that θ_k is uniformly distributed on the interval $[0, 2\pi]$ and that the receiver does not know what θ_k is.

We can use the representation of bandpass signals and noise in deriving the optimal receiver.

$$r(t) = \mathrm{Re}[u_k(t)e^{j\theta_k + j\omega_c t}] + n(t)$$

where

$$u_k(t) = a_k(t)\cos\beta_k(t) + ja_k(t)\sin\beta_k(t)$$

and $j = \sqrt{-1}$. Assuming the noise is also narrow band we can express the noise as

$$n(t) = n_c(t)\cos\omega_c t - n_s(t)\sin\omega_c t.$$

Then the lowpass representation of the received signal becomes

$$\tilde{r}(t) = u_k(t)e^{j\theta_k} + z(t)$$

where

$$z(t) = n_c(t) + jn_s(t).$$

Now assume that $z(t)$ is a Gaussian process with covariance function $K(s,t)$ which has eigenfunctions $\varphi_l(t)$ and eigenvalues λ_l. Then we can express the received lowpass signal as

$$\begin{aligned}\tilde{r}(t) &= \sum_{l=0}^{\infty}(u_{k,l}e^{j\theta_k} + z_l)\varphi_l(t)\\ &= \sum_{l=0}^{\infty}\tilde{r}_l\varphi_l(t)\end{aligned}$$

where

$$u_{k,l} = \int u_k(t)\varphi_l^*(t)dt$$

$$z_l = \int z(t)\varphi_l^*(t)dt$$

and z_l is a complex Gaussian random variable with mean zero and with

$$\begin{aligned}E[\mathrm{Re}(z_l)^2] &= \lambda_l\\ E[\mathrm{Im}(z_l)^2] &= \lambda_l\\ E[\mathrm{Re}(z_l)\mathrm{Im}(z_l)] &= 0.\end{aligned}$$

We can now calculate the probability density of $\tilde{\mathbf{r}} = (\tilde{r}_1, \ldots, \tilde{r}_N)$ conditioned on the value of θ_k. Let $\tilde{r}_l = u_{k,l}e^{j\theta_k} + z_l$ then

$$p_k(\tilde{r}_l|\theta_k) = \frac{1}{2\pi\lambda_l}\exp\left\{-\frac{1}{2\lambda_l}|\tilde{r}_l - e^{j\theta_k}u_{k,l}|^2\right\}$$

$$\begin{aligned}p_k(\tilde{\mathbf{r}}|\theta_k) &= \frac{1}{\prod_{l=1}^{N}2\pi\lambda_l}\exp\left\{-\frac{1}{2}\sum_{l=1}^{N}\frac{|\tilde{r}_l - e^{j\theta_k}u_{k,l}|^2}{\lambda_l}\right\}\\ &= \frac{1}{\prod_{l=1}^{N}2\pi\lambda_l}\exp\left\{-\frac{1}{2}\sum_{l=1}^{N}\frac{|\tilde{r}_l|^2 + |u_{k,l}|^2 - \tilde{r}_l e^{-j\theta_k}u_{k,l}^* - \tilde{r}_l^* e^{j\theta_k}u_{k,l}}{\lambda_l}\right\}\\ &= \prod_{l=1}^{N}(2\pi\lambda_l)^{-1}\exp\left\{-\frac{1}{2}\sum_{l=1}^{N}\frac{|\tilde{r}_l|^2 + |u_{k,l}|^2 - 2\mathrm{Re}(\tilde{r}_l e^{-j\theta_k}u_{k,l}^*)}{\lambda_l}\right\}\end{aligned}$$

$$= \prod_{l=1}^{N}(2\pi\lambda_l)^{-1}\exp\left\{-\frac{1}{2}\left[\sum_{l=1}^{N}\frac{|\tilde{r}_l|^2+|u_{k,l}|^2}{\lambda_l}\right]+\sum_{l=1}^{N}\frac{\mathrm{Re}(\tilde{r}_l u_{k,l}^*)\cos\theta_k-\mathrm{Im}(\tilde{r}_l u_{k,l}^*)\sin\theta_k}{\lambda_l}\right\}$$

$$= \prod_{l=1}^{N}(2\pi\lambda_l)^{-1}\exp\left\{-\frac{1}{2}\left[\sum_{l=1}^{N}\frac{|\tilde{r}_l|^2+|u_{k,l}|^2}{\lambda_l}\right]+\left(\left|\sum_{l=1}^{N}\frac{(\tilde{r}_l u_{k,l}^*)}{\lambda_l}\right|\cos(\theta_k+\psi)\right)\right\}.$$

The joint density given signal k transmitted is then obtained by averaging with respect to θ_k. The joint density given signal k transmitted is then

$$p_k(\tilde{r}_1,\ldots,\tilde{r}_N)=\int_{\theta=0}^{2\pi}\frac{1}{2\pi}p_k(\tilde{r}_1,\ldots,\tilde{r}_N|\theta_k)d\theta_k$$

$$=\int_{\theta=0}^{2\pi}\frac{1}{2\pi}\prod_{l=1}^{N}(2\pi\lambda_l)^{-1}\exp\left\{-\frac{1}{2}\left[\sum_{l=1}^{N}\frac{|\tilde{r}_l|^2+|u_{k,l}|^2}{\lambda_l}\right]+\left(\left|\sum_{l=1}^{N}\frac{(\tilde{r}_l u_{k,l}^*)}{\lambda_l}\right|\cos(\theta_k+\psi)\right)\right\}d\theta$$

$$=\prod_{l=1}^{N}(2\pi\lambda_l)^{-1}\exp\left\{-\frac{1}{2}\sum_{l=1}^{N}\frac{|\tilde{r}_l|^2+|u_{k,l}|^2}{\lambda_l}\right\}\int_{\theta=0}^{2\pi}\frac{1}{2\pi}\exp\left\{\left(\left|\sum_{l=1}^{N}\frac{(\tilde{r}_l u_{k,l}^*)}{\lambda_l}\right|\right)\cos\theta\right\}d\theta$$

$$=\prod_{l=1}^{N}(2\pi\lambda_l)^{-1}\exp\left\{-\frac{1}{2}\sum_{l=1}^{N}\frac{|\tilde{r}_l|^2+|u_{k,l}|^2}{\lambda_l}\right\}I_0\left(\left|\sum_{l=1}^{N}\frac{(\tilde{r}_l u_{k,l}^*)}{\lambda_l}\right|\right)$$

where

$$I_0(x)\triangleq\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\exp\{x\cos\theta\}\,d\theta$$

is the modified Bessel function of order 0. Now let us calculate the likelihood ratio between hypothesis H_k and the hypothesis that no signal was present.

$$\Lambda_k(N)=\frac{p_k(\tilde{r}_1,\ldots,\tilde{r}_N)}{p_0(\tilde{r}_1,\ldots,\tilde{r}_N)} = \frac{\prod_{l=1}^{N}(2\pi\lambda_l)^{-1}\exp\left\{-\frac{1}{2}\sum_{l=1}^{N}\frac{|\tilde{r}|^2+|u_{k,l}|^2}{\lambda_l}\right\}I_0\left(|\sum_{l=1}^{N}\frac{\tilde{r}u_{k,l}^*}{\lambda_l}|\right)}{\prod_{l=1}^{N}(2\pi\lambda_l)^{-1}\exp\{-\frac{1}{2}\sum_{l=1}^{N}\frac{|\tilde{r}|^2}{\lambda_l}\}}$$

$$= \exp\left\{-\frac{1}{2}\sum_{l=0}^{N}\frac{u_{k,l}u_{k,l}^*}{\lambda_l}\right\}I_0\left(\left|\sum_{l=1}^{N}\frac{\tilde{r}_l u_{k,l}^*}{\lambda_l}\right|\right).$$

Now

$$\lim_{N\to\infty}\sum_{l=1}^{N}\frac{u_{k,l}u_{k,l}^*}{\lambda_l}=\int u_k(t)q_k^*(t)dt$$

where

$$q_k(t)=\lim_{N\to\infty}\sum_{l=1}^{N}\frac{u_{k,l}}{\lambda_l}\varphi_l(t)$$

is the solution of the integral equation

$$u_k(s)=\int K(s,t)q_k(t)dt.$$

Similarly

$$\sum_{l=1}^{N}\frac{\tilde{r}_l u_{k,l}^*}{\lambda_l}=\int\tilde{r}(t)q_k^*(t)dt.$$

Thus the optimal receiver computes the following likelihood ratios

$$\Lambda_{k,0}(\tilde{r}(t))=\lim_{N\to\infty}(r(t))=\exp\{-\frac{1}{2}\int u_k(t)q_k^*(t)dt\}I_0\left(|\int\tilde{r}(t)q_k^*(t)dt|\right)$$

and chooses k for which $\Lambda_{k,0}$ is maximum.

Special Case: White Gaussian Noise

In this case the intergal equation is easily solved:

$$q_k(t)=\frac{2}{N_0}u_k(t)$$

so that

$$\Lambda_{k,0}(\tilde{r}(t))=\exp\{-\frac{1}{N_0}\int|u_k(t)|^2dt\}I_0\left(\frac{2}{N_0}|\int\tilde{r}(t)u_k^*(t)dt|\right).$$

For equi-energy signals this reduces to choosing k that maximizes

$$I_0\left(\frac{2}{N_0}|\int\tilde{r}(t)u_k^*(t)dt|\right)$$

and since $I_0(x)$ is an increasing function of x the optimal receiver chooses k to maximize

$$|\int\tilde{r}(t)u_k^*(t)dt|^2=(\mathrm{Re}\int\tilde{r}(t)u_k^*(t)dt)^2+(\mathrm{Im}\int\tilde{r}(t)u_k^*(t)dt)^2.$$

Now note that

$$r(t)=\mathrm{Re}\left[\tilde{r}(t)e^{j\omega_c t}\right]$$

and consider the following integral

$$\int r(t)a_k(t)\cos(\omega_c t+\beta_k(t))dt=\int\mathrm{Re}[\tilde{r}(t)e^{j\omega_c t}]\mathrm{Re}[u_k(t)e^{j\omega_c t}]dt.$$

Since (can you show this) for any two complex numbers a and b

$$\mathrm{Re}[a]\mathrm{Re}[b]=\frac{1}{2}\mathrm{Re}[ab^*]+\frac{1}{2}\mathrm{Re}[ab]$$

the above integral becomes

$$\int r(t)a_k(t)\cos(\omega_c t+\beta_k(t))dt=\frac{1}{2}\int\mathrm{Re}(\tilde{r}(t)(u_k^*(t))dt+\frac{1}{2}\int\mathrm{Re}(\tilde{r}(t)(u_k(t)e^{j2\omega_c t})dt.$$

That the second term is zero is due to the fact that both $\tilde{r}$ and u_k are lowpass processes. Thus

$$\int r(t)a_k(t)\cos(\omega_c t+\beta_k(t))dt=\frac{1}{2}\int\mathrm{Re}(\tilde{r}(t)(u_k^*(t))dt.$$

Similarly

$$\int r(t)a_k(t)\sin(\omega_c t+\beta_k(t))dt = \frac{1}{2}\int \mathrm{Im}(\tilde{r}(t)(u_k^*(t))dt.$$

Thus an equivalent form of the optimal receiver computes the following for each k

$$(\int r(t)a_k(t)\cos(\omega_c t+\beta_k(t))dt)^2 + (\int r(t)a_k(t)\sin(\omega_c t+\beta_k(t))dt)^2$$

and decides that signal k was transmitted if k maximizes the above expression.

Receiver Block Diagram

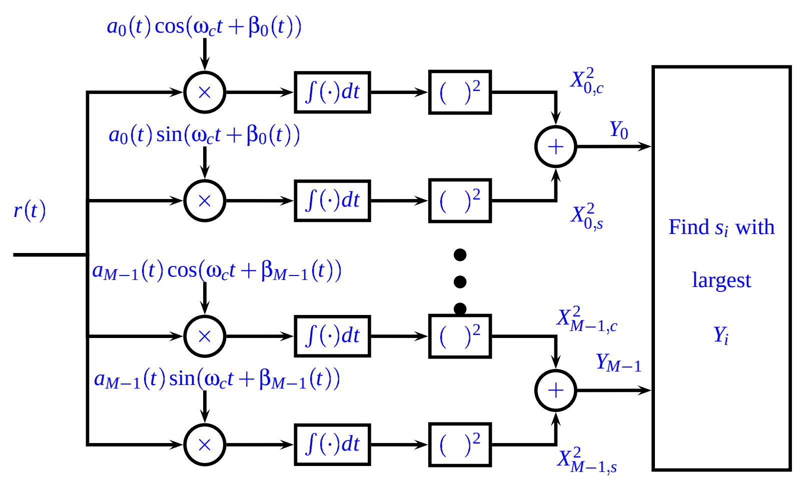


Figure 24: Optimum Receiver in Additive White Gaussian Noise

Performance in AWGN

Consider a binary communication system with noncoherent reception. Assume the two transmitted signals are

$$s_k(t) = a_k(t)\cos(\omega_c t+\beta_k(t)), \quad k=1,2$$

with

$$\frac{1}{2E}\int s_0(t)s_1(t)dt = \rho_{0,1} = \rho$$

Let H_1 denote the event that signal s_1 is transmitted and H_2 the event that s_2 is transmitted. The received signal differs from the transmitted signal in that there is a random phase term included and because of the noise. If s_1 is transmitted the received signal then is

$$r(t) = a_1(t)\cos(\omega_c t+\beta_1(t)+\theta_1)+n(t)$$

where $n(t)$ is a white Gaussian noise process. If s_2 is transmitted the received signal then is

$$r(t) = a_2(t)\cos(\omega_c t+\beta_2(t)+\theta_2)+n(t).$$

We would like to compute the error probability of the optimal receiver. The optimal receiver processes the received signal by correlating with two signals and then sums the squares. That

is the receiver first computes

$$X_{k,c} \triangleq \int r(t)a_k(t)\cos(\omega_c t+\beta_k(t))dt$$

and

$$X_{k,s} \triangleq \int r(t)a_k(t)\sin(\omega_c t+\beta_k(t))dt$$

Then

$$X_k = X_{k,c}^2 + X_{k,s}^2.$$

The receiver decides signal 2 was transmitted if $X_2 \geq X_1$ and otherwise decides s_1 transmitted. The probability of error given signal s_2 is transmitted is then

$$P\{error\ |H_1\} = P\{X_2 \geq X_1|H_1\}$$

To calculate the error probability we need to know the density of X_k. It is easy to see that $X_{k,c}$ and $X_{k,s}$ are a Gaussian random variables with mean

$$E[X_{k,c}|H_m,\theta_m] = \frac{1}{2}\rho_{k,m}\cos\theta_m$$

$$E[X_{k,c}|H_m,\theta_m] = \frac{1}{2}\rho_{k,m}\sin\theta_m$$

and variance

$$\mathrm{Var}[X_{k,c}|H_m,\theta_m] = \frac{1}{4}N_0E$$

where E is the energy of the transmitted signal. The density of X_k given H_m can then be calculated in a straightforward manner as

$$p_m(x_k) = \frac{1}{2\sigma^2}\exp\left\{-\frac{(\mu^2+x_k)}{2\sigma^2}\right\} I_0\left(\frac{\sqrt{x_k}\mu}{\sigma^2}\right), \quad x > 0$$

where

$$\begin{aligned} \mu &\triangleq \mu_c^2+\mu_s^2 \\ \mu_c &\triangleq \frac{1}{2}\rho_{k,m}\cos\theta_k \\ \mu_s &\triangleq \frac{1}{2}\rho_{k,m}\sin\theta_k \\ \sigma^2 &\triangleq \frac{1}{4}N_0E \end{aligned}$$

Involved calculation then yields

$$P\{\text{error } |H_1\} = P\{X_2 \geq X_1|H_1\} = Q(a,b) - \frac{1}{2}\exp\{(a^2+b^2)/2\}I_0(ab)$$

where

$$\begin{aligned} a &= \sqrt{\frac{E}{2N_0}(1-\sqrt{1-|\rho|^2})} \\ b &= \sqrt{\frac{E}{2N_0}(1+\sqrt{1-|\rho|^2})} \end{aligned}$$

and

$$Q(a,b) = \int_{b^2/2}^{\infty} \exp\{-(\frac{a^2}{2}+x)\}I_0(\sqrt{2x}a)dx$$

and is called Marcum's Q function.

Bit Error Probability of Nonorthogonal Signals

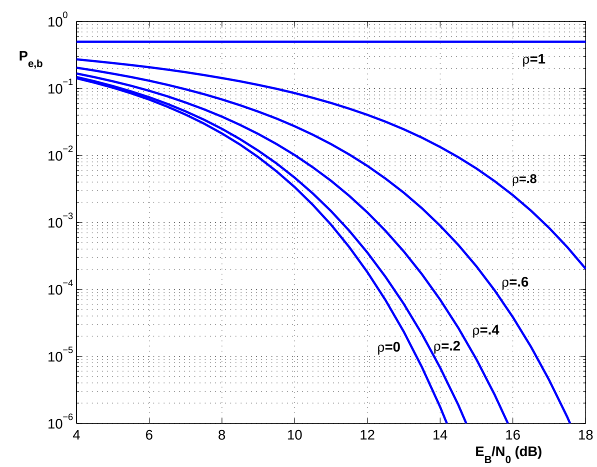


Figure 25: Performance of Nonorthogonal Signals with Noncoherent Demodulation

Error Probability for M-orthogonal Signals: Noncoherent reception in AWGN

Preliminaries: First we derive the density for the sum of the squares of two Gaussian random variables. Let

$$\begin{aligned} X_c &\sim N(\mu_c, \sigma^2) \\ X_s &\sim N(\mu_s, \sigma^2) \end{aligned}$$

with X_c, X_s independent. Let $\mu^2 = \mu_c^2 + \mu_s^2$ and

$$Y = X_c^2 + X_s^2.$$

Then

$$\begin{aligned} P\{Y \le y\} &= \iint_{x_c^2+x_s^2 \le y} \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}\left[(x_c-\mu_c)^2 + (x_s-\mu_s)^2\right]\right\} dx_c dx_s \\ &= \iint_{x_c^2+x_s^2 \le y} \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}[x_c^2 + x_s^2 - 2(x_c\mu_c + x_s\mu_s) + \mu^2]\right\} dx_c dx_s \end{aligned}$$

$$= \iint_{x_c^2+x_s^2 \le y} \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}\left[x_c^2 + x_s^2 - 2\mu\sqrt{x_c^2+x_s^2}\cos(\phi+\gamma) + \mu^2\right]\right\} dx_c dx_s$$

where $\phi = \tan^{-1}\frac{x_s}{x_c}$ and $\gamma = \tan^{-1}\left(\frac{-\mu_s}{\mu_c}\right)$.

$$\begin{aligned} &= \int_{r^2 \le \sqrt{y}} \int_{\phi=0}^{2\pi} \frac{r}{2\pi\sigma^2} \exp\left\{-[\frac{r^2}{2\sigma^2} - \frac{\mu r}{\sigma^2}\cos(\phi+\gamma) + \frac{\mu^2}{2\sigma^2}]\right\} dr d\phi \\ &= \int_{r \le \sqrt{y}} \frac{r}{\sigma^2} \exp\left\{\frac{-r^2}{2\sigma^2}\right\} e^{-\mu^2/2\sigma^2} \underbrace{\frac{1}{2\pi}\int_{\phi=0}^{2\pi} \exp\left\{\frac{\mu r}{\sigma^2}\cos(\phi+\gamma)\right\} d\phi}_{I_0\left(\frac{\mu r}{\sigma^2}\right)} dr \end{aligned}$$

$$\begin{aligned} x_c\mu_c + x_s\mu_s &= \beta\cos(\phi+\gamma) \\ &= \beta\left[\cos\phi\cos\gamma - \sin\phi\sin\gamma\right] \end{aligned}$$

$$\phi = \tan^{-1}\frac{x_s}{x_c} \qquad \begin{aligned} \tan\phi &= \tfrac{x_s}{x_c} \\ \cos\phi &= \tfrac{x_c}{\sqrt{x_c^2+x_s^2}} \\ \sin\phi &= \tfrac{x_s}{\sqrt{x_0^2+x_s^2}} \end{aligned}$$

$$\begin{aligned} \cos\gamma &= \frac{\mu_c}{\sqrt{\mu_c^2+\mu_s^2}} \quad \sin\gamma = \frac{-\mu_s}{\sqrt{\mu_c^2+\mu_s^2}} \\ \beta &= \sqrt{\mu_c^2+\mu_s^2}\sqrt{x_c^2+x_s^2} \\ x_c\mu_c + x_s\mu_s &= \sqrt{\mu_c^2+\mu_s^2}\sqrt{x_c^2+x_s^2}\left[\cos(\phi+\gamma)\right] \\ &\phi = \tan^{-1}\left(\frac{x_s}{x_c}\right) \\ &\gamma = \tan^{-1}\left(\frac{-\mu_s}{\mu_c}\right) \end{aligned}$$

$$P\{Y \le y\} = \int_{r<\sqrt{y}} \frac{r}{\sigma^2} \exp\left\{-\frac{r^2+\mu^2}{2\sigma^2}\right\} I_0\left(\frac{\mu r}{\sigma^2}\right) dr$$

Let $u = r^2$ then $0 \le r \le \sqrt{y}$ is equivalent to $u \le y$. Also $du = 2rdr$.

$$\begin{aligned} P\{Y \le y\} &= \int_{u \le y} \frac{1}{2\sigma^2} \exp\left\{-\frac{u+\mu^2}{2\sigma^2}\right\} I_0\left(\frac{\mu\sqrt{u}}{\sigma^2}\right) du \\ f_Y(y) &= \frac{1}{2\sigma^2} \exp\left\{-\frac{y+\mu^2}{2\sigma^2}\right\} I_0\left(\frac{\mu\sqrt{y}}{\sigma^2}\right) \end{aligned}$$

A change of variables makes for a cleaner expression: Let $W = Y/(2\sigma^2)$. Then $f_W(w) = 2\sigma^2 f_Y(2\sigma^2 w)$

$$f_W(w) = \exp\{-(w+\Gamma)\} I_0\left(\sqrt{4\Gamma w}\right)$$

where $\Gamma = \mu^2/(2\sigma^2)$. (If the receiver does this normalization then it must know the power density of the noise). Now let $Z = \sqrt{Y}$. Then

$$\begin{aligned} P\{Z \le z\} &= P\left\{\sqrt{Y} \le z\right\} = P\left\{Y \le z^2\right\} \\ F_Z(z) &= F_Y(z^2) \\ f_Z(z) &= f_Y(z^2)(2z) \\ &= \frac{z}{\sigma^2} \exp\left\{-\frac{z^2+\mu^2}{2\sigma^2}\right\} I_0\left(\frac{\mu z}{\sigma^2}\right) \end{aligned}$$

$$\mu = 0 \quad \Rightarrow \quad f_Y(y) = \frac{1}{2\sigma^2}\, e^{-y/2\sigma^2}$$
$$f_Z(z) = \frac{z}{\sigma^2}\, e^{-z^2/2\sigma^2}$$

Error Probability

$$\begin{aligned} X_{jc} &= E\,\delta_{ij}\cos\phi_i + n_c \sim N\left(E\delta_{ij}\cos\phi_i, \frac{N_0E}{2}\right) \\ X_{js} &= E\delta_{ij}\sin\phi_i + n, \sim N\left(E\delta_{ij}\sin\phi_i, \frac{N_0E}{2}\right) \\ j &= 0,2,\ldots,M-1 \end{aligned}$$

Let $Z_j = \sqrt{X_{jc}^2 + X_{js}^2}$. Then we need to determine the probability that Z_0 which corresponds to nonzero mean random variables is less than $Z_1, Z_2, ..., Z_{M-1}$ which correspond to zero mean random variables.

$$\begin{aligned} P_{c,i} &= P\{Z_j \le Z_i, j \ne i \mid s_i \text{ transmitted}\} \\ &= E[P\{Z_j < Z_i, j \ne i \mid Z_i, s_i \text{ trans}\}] \\ &= E[\prod_{j\ne i} P\{Z_j < Z_i \mid s_i \text{ trans}, Z_i\}] \\ &= \int p(z_i)[P\{Z_j \le z_i\}]^{M-1}\, dz_i \end{aligned}$$

Doing a change of variables ($\nu = \frac{u^2}{2\sigma^2}$, $d\nu = \frac{udu}{\sigma^2}$) we obtain

$$\begin{aligned} P\{Z_j \le z_i\} &= \int_{u\le z_i} \frac{u}{\sigma^2}\, e^{-u^2/2\sigma^2} du = \int_{\nu \le z_i^2/2\sigma^2} e^{-\nu} d\nu \\ &= 1 - e^{-z_i^2/2\sigma^2} \\ P_{c,i} &= \int_0^\infty p(z_i)[1 - e^{-z_i^2/2\sigma^2}]^{M-1}\, dz_i \\ &= \int_0^\infty p(z_i)\left(\sum_{l=0}^{M-1}(-1)^l \binom{M-1}{l} e^{-lz_i^2/2\sigma^2}\right) dz_i \\ &= \sum_{l=0}^{M-1}(-1)^l \binom{M-1}{l} \int_0^\infty p(z_i) e^{-lz_i^2/2\sigma^2} dz_i \end{aligned}$$

$$\begin{aligned} \int_0^\infty p(z_i)e^{-lz_i^2/2\sigma^2} dz_i &= \int_0^\infty \frac{z_i}{\sigma^2}\, e^{\frac{-(z_i^2+\mu^2)}{2\sigma^2}}\, I_0\left(\frac{\mu z_i}{\sigma^2}\right) e^{-lz_i^2/2\sigma^2} dz_i \\ &= e^{-\mu^2/2\sigma^2} \int_0^\infty \frac{z_i}{\sigma^2}\, e^{-(l+1)z_i^2/2\sigma^2}\, I_0\left(\frac{\mu z_i}{\sigma^2}\right) dz_i \end{aligned}$$

Do another change of variables, ($w_i = \sqrt{l+1}\, z_i$, $dw_i = \sqrt{l+1}\, dz_i$) we get

$$\int_0^\infty p(z_i)e^{-lz_i^2/2\sigma^2} dz_i = e^{-\mu^2/2\sigma^2} \int_0^\infty \frac{w_i}{\sigma^2\sqrt{l+1}} e^{-w_i^2/(2\sigma^2)} I_0\left(\frac{\mu w_i}{\sqrt{l+1}\,\sigma^2}\right) \frac{dw_i}{\sqrt{l+1}}$$

$$= e^{-\mu^2/2\sigma^2} \frac{1}{(l+1)} \int \frac{w_i}{\sigma^2}\, e^{-\frac{w_i^2}{2\sigma^2}}\, I_0\left(\frac{\mu w_i}{\sqrt{l+1}\,\sigma^2}\right) dw_i$$

Let $\hat{\mu} = \frac{\mu}{\sqrt{l+1}}$. Then

$$\begin{aligned} \int_0^\infty p(z_i)e^{-lz_i^2/2\sigma^2} dz_i &= \frac{e^{-\mu^2/2\sigma^2} e^{\hat{\mu}^2/2\sigma^2}}{l+1} \underbrace{\int \frac{w_i}{\sigma^2}\, e^{-\left(\frac{w_i^2+\hat{\mu}^2}{2\sigma^2}\right)} I_0\left(\frac{\hat{\mu} w_i}{\sigma^2}\right) dw_i}_{=1} \\ &= \frac{\exp\left\{\frac{\hat{\mu}^2-\mu^2}{2\sigma^2}\right\}}{(l+1)} = \frac{\exp\left\{\frac{\frac{\mu^2}{l+1}-\mu^2}{2\sigma^2}\right\}}{l+1} \\ &= \exp\left\{-\frac{l\mu^2}{2(l+1)\sigma^2}\right\} \frac{1}{l+1}. \end{aligned}$$

Substituting this into the expression for the probability of correct, we obtain

$$P_{c,i} = \sum_{l=0}^{M-1}(-1)^l \binom{M-1}{l} \frac{\exp\left\{\frac{-l\mu^2}{2(l+1)\sigma^2}\right\}}{(l+1)}$$

where

$$\frac{\mu^2}{2\sigma^2} = \frac{E^2}{2\frac{N_0 E}{2}} = \frac{E}{N_0}.$$

Thus

$$\begin{aligned} P_{c,i} &= 1 + \sum_{l=1}^{M-1} \frac{(-1)^l \binom{M-1}{l}}{l+1} \exp\left\{-\frac{l}{(l+1)} \frac{E}{N_0}\right\} \\ P_{e,i} &= 1 - P_{c,i} = \sum_{l=1}^{M-1} \frac{(-1)^{l+1} \binom{M-1}{l}}{l+1} \exp\left\{-\frac{l}{l+1} \frac{E}{N_0}\right\} \end{aligned}$$

$$P_e = \frac{1}{M} \sum_{i=1}^{M-1} P_{e,i} = P_{e,i}$$

$$M \text{ signals} \Rightarrow \log_2 M \text{ bits}$$

The limiting behavior of the error probability for M-ary orthogonal signals with noncoherent demodulation is the same as the limiting performance of coherent demodulation. If $M = 2^k$

$$P_{e,b} = \frac{1}{2} \frac{M}{M-1} P_{e,i} \quad E_b = E/\log_2 M(R)$$

$$M = 2 \;\; P_{e,b} = P_{e,i} = \frac{1}{2} e^{-E/2N_0}$$

It can be shown that the asymptotic behavior of M orthogonal signals on a additive Gaussian noise channel with noncoherent reception is the same as with coherent reception. That is for $E_b/N_0 < \ln 2$ the error probability is 1 while for $E_b/N_0 > \ln 2$ the error probability is 0.

Error Probability for M orthogonal signals

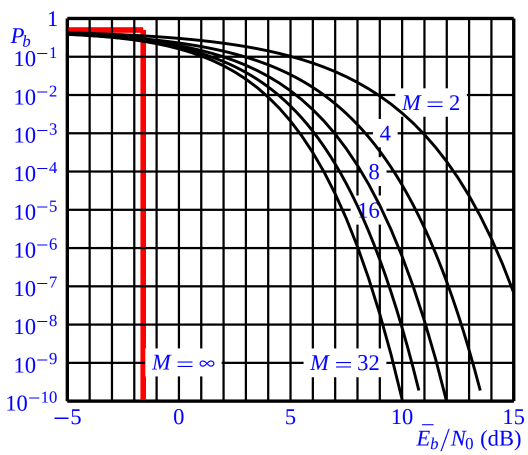


Figure 26: Bit error probability of M-ary orthogonal modulation in an additive white Gaussian noise channel with noncoherent demodulation

Error Estimates for Repetition Codes with Noncoherent Reception

Consider transmitting one of two codewords of length L using binary frequency shift keying (orthogonal) and noncoherent reception. The optimum receiver would compute the log-likelihood ratio and compare to zero (for equally likely codewords) to determine which of the two codewords was transmitted. Assume that the first codeword is the all zero vector (0,0,...,0) of length L and the other codeword is the all one vector (1,1,...,1) of length L. If the two codewords agreed in some positions then we would not need to process the received signal over the interval of time where they agreed since no information can be gained about which signal was transmitted from the received signal in that interval.

The transmitted signal would be

$$s_0(t) = \sum_{l=0}^{L-1} \sqrt{2P} \cos(\omega_0 t + \theta_{0,l}) p_T(t - lT)$$

or

$$s_1(t) = \sum_{l=0}^{L-1} \sqrt{2P}\cos(\omega_1 t + \theta_{1,l}) p_T(t - lT)$$

where $\theta_{i,l}$ are independent identically distributed random variables with uniform distribution unknown to the receiver. The received signal is

$$r(t) = \begin{cases} s_0(t) + n(t) & H_0 \text{ true} \\ s_1(t) + n(t) & H_1 \text{ true} \end{cases}$$

The receiver processes the signals using noncoherent matched filters; that is the received signal is multiplied by $\exp j\omega_0 t$ then passed through a filter with impulse response $p_T(t)$, sampled and then the magnitude is formed. Denote by $Y_{0,l}$ the output of the noncoherent matched filter

$$Y_{0,l} = |\int_{-\infty}^{\infty} r(s) p_T(s - lT) \exp\{j\omega_0(s - lT)\} ds|^2$$

and

$$Y_{1,l} = |\int_{-\infty}^{\infty} r(s) p_T(s - lT) \exp\{j\omega_1(s - lT)\} ds|^2$$

The statistics of $Y_{0,l}, Y_{1,l}$ were calculated in the previous section. Because of orthogonality of the received signals and that the noise is white the joint statistics of $\mathbf{Y} = (Y_{0,1}, \ldots, Y_{0,L}, Y_{1,1}, \ldots, Y_{1,L})$ factor (conditioned on either of the two hypotheses) as

$$p(y_{0,1}, \ldots, y_{0,L}, y_{1,1}, \ldots, y_{1,L}, |H_k) = \prod_{l=1}^{L} p(y_{0,l}|H_k) p(y_{1,l}|H_k) \quad k = 0, 1.$$

The log-likelihood ratio is then

$$\begin{aligned} \Lambda = \log \frac{p(\mathbf{y}|H_1)}{p(\mathbf{y}|H_0)} &= \log \prod_{l=1}^{L} \frac{p(y_{0,l}|H_1) p(y_{1,l}|H_1)}{p(y_{0,l}|H_0) p(y_{1,l}|H_0)} \\ &= \sum_{l=1}^{L} \log \frac{p(y_{0,l}|H_1) p(y_{1,l}|H_1)}{p(y_{0,l}|H_0) p(y_{1,l}|H_0)} \\ &= \sum_{l=1}^{L} \log\{p(y_{0,l}|H_1) p(y_{1,l}|H_1)\} - \log\{p(y_{0,l}|H_0) p(y_{1,l}|H_0)\} \end{aligned}$$

Substituting in the appropriate density functions yields

$$\Lambda = \sum_{l=1}^{L} \log[I_0(\frac{\mu\sqrt{y_{1,l}}}{\sigma^2})] - \log[I_0(\frac{\mu\sqrt{y_{0,l}}}{\sigma^2})]$$

The optimum receiver is thus

$$\sum_{l=1}^{L} \log[I_0(\frac{\mu\sqrt{y_{1,l}}}{\sigma^2})] \underset{H_0}{\overset{H_1}{\gtrless}} \sum_{l=1}^{L} \log[I_0(\frac{\mu\sqrt{y_{0,l}}}{\sigma^2})]$$

which is quite complex in implementation. It would be useful to have suboptimum receivers which are easier to implement but have nearly optimum performance. Before we suggest some suboptimal receivers is would be useful to get estimates of the performance of the optimum receiver. The error probability (given H_0 say) is easy to write down but hard to evaluate except for L small. It is

$$P_e = \int_{\mathbf{y}} I[R_1] p(\mathbf{y}|H_0) d\mathbf{y}$$

where $I[R_1]$ is the region where the log-likelihood ratio is positive. This is a $2L$ dimensional integral. The Chernoff bound to the error probability can be expressed as

$$P_e \le D^L$$

where

$$D = \int_{\mathbf{y}} \sqrt{p(\mathbf{y}|H_0) p(\mathbf{y}|H_1)} d\mathbf{y}.$$

For the additive white Gaussian channel

$$\begin{aligned} D &= \left[\int_0^{\infty} \frac{1}{2\sigma^2} \exp\{-\frac{1}{2}(\frac{y}{\sigma^2} + \frac{\mu^2}{\sigma^2})\} \sqrt{I_0(\frac{\mu\sqrt{y}}{\sigma^2})} dy\right]^2 \\ &= \left[\int_0^{\infty} \exp\{-(w + \Gamma/2)\} \sqrt{I_0(\sqrt{4\Gamma w})} dw\right]^2 \end{aligned}$$

and $\Gamma = E/N_0$. This is much more computationally attractive than the exact expression. An asymptotic form for low signal-to-noise ratios is not to difficult to compute.

$$D \approx 1 - \left(\frac{\Gamma}{2}\right)^2 \qquad \Gamma \text{ small.}$$

For hard decisions

$$D = 2\sqrt{p(1-p)}, \qquad p = \frac{1}{2} e^{-\frac{\Gamma}{2}}$$

which for low signal-to-noise ratios becomes

$$D \approx 1 - \left(\frac{\Gamma}{2\sqrt{2}}\right)^2 \qquad \Gamma \text{ small.}$$

Thus for low signal-to-noise ratios hard decisions cost $\sqrt{2}$=1.5dB.

Now let us consider some suboptimal receivers. First note that $\mu/\sigma^2 = 2/N_0$. So the argument to the Bessel function will be small if the noise power is large (low signal-to-noise ratio) and will be large if the noise power is small (high signal-to-noise ratio). Also note that (see Abramowitz and Stegun *Handbook of Mathematical Functions*)

$$\begin{aligned} I_0(x) &\approx 1+\frac{x^2}{4}, \quad x \text{ small} \\ \log I_0(x) &\approx \frac{x^2}{4}, \quad x \text{ small} \\ I_0(x) &\approx \frac{e^x}{\sqrt{2\pi x}}, \quad x \text{ large} \end{aligned}$$

Using the approximation for $\log I_0$ in the optimum decision rule we get

$$\begin{aligned} \sum_{l=1}^{L}\left(\frac{\mu^2 y_{1,l}}{4\sigma^4}\right) &\underset{H_0}{\overset{H_1}{\gtrless}} \sum_{l=1}^{L}\left(\frac{\mu^2 y_{0,l}}{4\sigma^4}\right) \\ \sum_{l=1}^{L} y_{1,l} &\underset{H_0}{\overset{H_1}{\gtrless}} \sum_{l=1}^{L} y_{0,l} \end{aligned}$$

So for small signal-to-noise ratios the optimum receiver is the square-law combining receiver.

Now consider when the argument to the Bessel function is large. The optimum decision rule

then becomes

$$\sum_{l=1}^{L}\left(\frac{\mu\sqrt{y_{1,l}}}{2\sigma^2}\right) - \log\sqrt{2\pi\mu\sqrt{y_{1,l}}/\sigma^2} - \left(\frac{\mu\sqrt{y_{0,l}}}{2\sigma^2}\right) + \log\sqrt{2\pi\mu\sqrt{y_{0,l}}/\sigma^2} \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

Let $w_{i,l} = y_{i,l}/(2\sigma^2)$ then the decision rule is

$$\sum_{l=1}^{L}\left(\sqrt{4\Gamma w_{1,l}}\right) - \log\sqrt{2\pi\sqrt{4\Gamma w_{1,l}}} - \left(\sqrt{4\Gamma w_{0,l}}\right) + \log\sqrt{2\pi\sqrt{4\Gamma w_{0,l}}} \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

Note that the average value of W given signal present is $\Gamma+1$ while the average value of W given no signal is 1. For very large Γ the terms $(\sqrt{4\Gamma w})$ dominates the terms of the form $\log\sqrt{2\pi\sqrt{4\Gamma w}}$ and thus the optimum decision rule is

$$\sum_{l=1}^{L}\left(\sqrt{w_{1,l}}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \sum_{l=1}^{L}\left(\sqrt{w_{0,l}}\right)$$

Thus the decision rule for very high signal-to-noise ratio is to add the square-roots of the noncoherent matched filter outputs.

It is of interest to analyze the performance of these suboptimal receivers. The receiver for very low signal-to-noise ratios is (relatively) easy to analyze. First let us normalize the density for

the sum of the squares of $2L$ random variables. Let

$$W_0 = \frac{1}{2\sigma^2}\sum_{l=1}^{L} X_{c,0,l}^2 + X_{s,0,l}^2$$

and similarly for W_1. Then the density of W is given by

$$\begin{aligned} f_{W_0}(w_0|H_0) &= \left(\frac{w_0}{\Gamma}\right)^{(L-1)/2}\exp\{-(w_0+\Gamma)\}I_{L-1}(\sqrt{4w_0\Gamma}) \quad w \geq 0 \\ f_{W_0}(w_0|H_1) &= \frac{w_0^{(L-1)}}{(L-1)!}\exp\{-w_0\} \quad w \geq 0 \end{aligned}$$

and similarly for W_1.

$$\begin{aligned} P_e &= 1 - P\{W_0 > W_1|H_0\} \\ &= 1 - \int_0^{\infty} f_{W_0}(w_0|H_0)P\{W_1 < w_0|H_0\}dw_0 \\ &= 1 - \int_0^{\infty} f_{W_0}(w_0|H_0)\left[1 - \sum_{m=0}^{L-1}\frac{w_0^m}{m!}e^{-w_0}\right]dw_0 \\ P_e &= \frac{1}{2}\exp\left\{-\frac{\Gamma}{2}\right\}\sum_{i=0}^{L-1}\frac{(\Gamma/2)^i}{i!(L+i-1)!}\sum_{j=i}^{L-1}\frac{(j+L-1)!}{(j-i)!2^{j+L-1}} \end{aligned}$$

For $L=1$ the above becomes

$$P_e = \frac{1}{2}e^{-\Gamma/2}$$

where $\Gamma = E/N_0$. The Chernoff bound can also be calculated for square-law combining.

Primer on sums of squares of Gaussian random variables

First we derive the density for the sum of the squares of two Gaussian random variables. Let

$$\begin{aligned} X_c &\sim N(\mu_c, \sigma^2) \\ X_s &\sim N(\mu_s, \sigma^2) \end{aligned}$$

with X_c, X_s independent. Let $\mu^2 = \mu_c^2 + \mu_s^2$ and

$$Y = X_c^2 + X_s^2.$$

Then

$$\begin{aligned} P\{Y \le y\} &= \iint\limits_{x_c^2+x_s^2 \le y} \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}\left[(x_c-\mu_c)^2 + (x_s-\mu_s)^2\right]\right\} dx_c dx_s \\ &= \iint\limits_{x_c^2+x_s^2 \le y} \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}[x_c^2 + x_s^2 - 2(x_c\mu_c + x_s\mu_s) + \mu^2]\right\} dx_c dx_s \\ &= \iint\limits_{x_c^2+x_s^2 \le y} \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}\left[x_c^2 + x_s^2 - 2\mu\sqrt{x_c^2+x_s^2}\right.\right. \end{aligned}$$

$$\left.\left.\cos(\phi+\gamma) + \mu^2\right]\right\} dx_c dx_s$$

where $\phi = \tan^{-1}\frac{x_s}{x_c}$ and $\gamma = \tan^{-1}\left(\frac{-\mu_s}{\mu_c}\right)$.

$$\begin{aligned} &= \int_{r^2 \le \sqrt{y}} \int_{\phi=0}^{2\pi} \frac{r}{2\pi\sigma^2} \exp\left\{-[\frac{r^2}{2\sigma^2} - \frac{\mu r}{\sigma^2}\cos(\phi+\gamma) + \frac{\mu^2}{2\sigma^2}]\right\} dr d\phi \\ &= \int_{r \le \sqrt{y}} \frac{r}{\sigma^2} \exp\left\{\frac{-r^2}{2\sigma^2}\right\} e^{-\mu^2/2\sigma^2} \underbrace{\frac{1}{2\pi}\int_{\phi=0}^{2\pi} \exp\left\{\frac{\mu r}{\sigma^2}\cos(\phi+\gamma)\right\} d\phi}_{I_0\left(\frac{\mu r}{\sigma^2}\right)} dr \end{aligned}$$

$$\begin{aligned} x_c\mu_c + x_s\mu_s &= \beta\cos(\phi+\gamma) \\ &= \beta\left[\cos\phi\cos\gamma - \sin\phi\sin\gamma\right] \end{aligned}$$

$$\phi = \tan^{-1}\frac{x_s}{x_c} \qquad \begin{aligned} \tan\phi &= \frac{x_s}{x_c} \\ \cos\phi &= \frac{x_c}{\sqrt{x_c^2+x_s^2}} \\ \sin\phi &= \frac{x_s}{\sqrt{x_0^2+x_s^2}} \end{aligned}$$

$$\cos\gamma = \frac{\mu_c}{\sqrt{\mu_c^2+\mu_s^2}} \qquad \sin\gamma = \frac{-\mu_s}{\sqrt{\mu_c^2+\mu_s^2}}$$

$$\begin{aligned} \beta &= \sqrt{\mu_c^2+\mu_s^2}\sqrt{x_c^2+x_s^2} \\ x_c\mu_c + x_s\mu_s &= \sqrt{\mu_c^2+\mu_s^2}\sqrt{x_c^2+x_s^2}\left[\cos(\phi+\gamma)\right] \end{aligned}$$

$$\phi = \tan^{-1}\left(\frac{x_s}{x_c}\right)$$

$$\gamma = \tan^{-1}\left(\frac{-\mu_s}{\mu_c}\right)$$

$$P\{Y \le y\} = \int_{r<\sqrt{y}} \frac{r}{\sigma^2} \exp\left\{-\frac{r^2+\mu^2}{2\sigma^2}\right\} I_0\left(\frac{\mu r}{\sigma^2}\right) dr$$

Let $u = r^2$ then $0 \le r \le \sqrt{y}$ is equivalent to $u \le y$. Also $du = 2rdr$.

$$\begin{aligned} P\{Y \le y\} &= \int_{u \le y} \frac{1}{2\sigma^2} \exp\left\{-\frac{u+\mu^2}{2\sigma^2}\right\} I_0\left(\frac{\mu\sqrt{u}}{\sigma^2}\right) du \\ f_Y(y) &= \frac{1}{2\sigma^2} \exp\left\{-\frac{y+\mu^2}{2\sigma^2}\right\} I_0\left(\frac{\mu\sqrt{y}}{\sigma^2}\right) \end{aligned}$$

A change of variables makes for a cleaner expression: Let $W = Y/(2\sigma^2)$. Then

$f_W(w) = 2\sigma^2 f_Y(2\sigma^2 w)$

$$f_W(w) = \exp\{-(w+\Gamma)\} I_0\left(\sqrt{4\Gamma w}\right)$$

where $\Gamma = \mu^2/(2\sigma^2)$. (If the receiver does this normalization then it must know the power density of the noise). Now let $Z = \sqrt{Y}$. Then

$$\begin{aligned} P\{Z \le z\} &= P\left\{\sqrt{Y} \le z\right\} = P\left\{Y \le z^2\right\} \\ F_Z(z) &= F_Y(z^2) \\ f_Z(z) &= f_Y(z^2)(2z) \\ &= \frac{z}{\sigma^2} \exp\left\{-\frac{z^2+\mu^2}{2\sigma^2}\right\} I_0\left(\frac{\mu z}{\sigma^2}\right) \end{aligned}$$

$$\mu = 0 \quad \Rightarrow \quad f_Y(y) = \frac{1}{2\sigma^2} e^{-y/2\sigma^2}$$

$$f_Z(z) = \frac{z}{\sigma^2} e^{-z^2/2\sigma^2}$$

Using the fact that a density must integrate to one we can derive an useful integral.

$$\int_0^\infty \frac{r}{\sigma^2} \exp\{-r^2/2\sigma^2\} \exp\{-\alpha r^2\} I_0(r\beta) dr = \frac{1}{1+2\sigma^2\alpha} \exp\{\frac{\sigma^2\beta^2}{1+2\sigma^2\alpha}\}$$

Generalization:

$$\begin{aligned} X_{c,i} &\sim N(\mu_{c,i},\sigma^2) \qquad i=1,2,\ldots,L \\ X_{s,i} &\sim N(\mu_{s,i},\sigma^2) \qquad i=1,2,\ldots,L \end{aligned}$$

with $X_{c,i}, X_{s,i}$ independent. Let $\Lambda=\sum_{i=1}^{L}\mu_{c,i}^2+\mu_{s,i}^2$ and

$$Y=\sum_{i=1}^{L} X_{c,i}^2+X_{s,i}^2.$$

Then

$$f_Y(y)=\frac{1}{2\sigma^2}\exp\{-\left(\frac{y+\Lambda}{2\sigma^2}\right)\}\left(\frac{y}{\Lambda}\right)^{(L-1)/2} I_{L-1}\left(\frac{\sqrt{y\Lambda}}{\sigma^2}\right)$$

$$F_Y(y)=1-Q_L\left(\frac{\sqrt{\Lambda}}{\sigma},\frac{\sqrt{y}}{\sigma}\right)$$

where

$$Q_L(a,b)=Q(a,b)+\exp\{(a^2+b^2)/2\}\sum_{k=1}^{L-1}\left(\frac{b}{a}\right)^k I_k(ab)$$

and

$$Q(a,b)=\exp\{-(a^2+b^2)/2\}\sum_{k=1}^{\infty}\left(\frac{b}{a}\right)^k I_k(ab)$$

For $\Lambda=0$

$$\begin{aligned} f_Y(y) &= \frac{1}{2\sigma^2}\exp\{-\left(\frac{y}{2\sigma^2}\right)\}\left(\frac{y}{2\sigma^2}\right)^{(L-1)}\frac{1}{(L-1)!} \\ F_Y(y) &= 1-\exp\{-\left(\frac{y}{2\sigma^2}\right)\}\sum_{k=0}^{L-1}\frac{1}{k!}\left(\frac{y}{2\sigma^2}\right)^k \end{aligned}$$

Let $Z=\sqrt{Y}$ then

$$\begin{aligned} f_Z(z) &= \frac{z^L}{\sigma^2\Lambda^{(L-1)/2}}\exp\{-\left(\frac{z^2+\Lambda}{2\sigma^2}\right)\}I_{L-1}\left(\frac{z\sqrt{\Lambda}}{\sigma^2}\right) \\ F_Z(z) &= 1-Q_L\left(\frac{\sqrt{\Lambda}}{\sigma},\frac{z}{\sigma}\right) \end{aligned}$$

For $\Lambda=0$ we obtain

$$\begin{aligned} f_Z(z) &= \frac{z^{2L-1}}{2^{L-1}\sigma^{2L}(L-1)!}\exp\{-\frac{z^2}{2\sigma^2}\} \\ F_Z(z) &= 1-\exp\{-\frac{z^2}{2\sigma^2}\}\sum_{l=0}^{L-1}\frac{(z^2/(2\sigma^2))^l}{l!} \end{aligned}$$

Consider two random variables Z_1 and Z_2 where Z_1 has distribution given above with $\Lambda_1=0$

and with different variances σ_1 and σ_2. Assume that they are independent. We wish to determine the probability that $Z_1>Z_2$.

$$\begin{aligned} P\{Z_1<Z_2\} &= \int_{z_2=0}^{\infty} P\{Z_1\le z_2\} f_{Z_2}(z_2)dz_2 \\ &= \int_{z_2=0}^{\infty} F_{Z_1}(z_2) f_{Z_2}(z_2)dz_2 \\ &= \int_{z_2=0}^{\infty}\left[1-\exp\{-\frac{z_2^2}{2\sigma_1^2}\}\sum_{l=0}^{L-1}\frac{(z_2^2/(2\sigma_1^2))^l}{l!}\right] f_{Z_2}(z_2)dz_2 \\ &= \int_{z_2=0}^{\infty}\left[1-\exp\{-\frac{z_2^2}{2\sigma_1^2}\}\sum_{l=0}^{L-1}\frac{(z_2^2/(2\sigma_1^2))^l}{l!}\right] \\ &\qquad \frac{z_2^L}{\sigma_2^2\Lambda^{(L-1)/2}}\exp\{-\frac{z_2^2+\Lambda}{2\sigma_2^2}\}I_{L-1}\left(\frac{z_2\sqrt{\Lambda}}{\sigma_2^2}\right)dz_2 \\ &= 1-\sum_{l=0}^{L-1}\frac{1}{(2\sigma_1^2)^l l!}\exp\{-\left(\frac{\Lambda}{2\sigma_2^2}\right)\}\int_{z=0}^{\infty}\exp\{-\frac{z^2}{2\sigma_1^2}-\frac{z^2}{2\sigma_2^2}\} \\ &\qquad \frac{z^{L+2l}}{\sigma_2^2\Lambda^{(L-1)/2}}I_{L-1}\left(\frac{z\sqrt{\Lambda}}{\sigma_2^2}\right)dz \\ &= 1-\sum_{l=0}^{L-1}\frac{1}{(2\sigma_1^2)^l l!\sigma_2^2\Lambda^{(L-1)/2}}\exp\{-\left(\frac{\Lambda}{2\sigma_2^2}\right)\}\int_{z=0}^{\infty}e^{-\alpha^2z^2}z^{L+2l}I_{L-1}(\gamma z)dz \end{aligned}$$

where

$$\begin{aligned} \alpha^2 &= \frac{1}{2\sigma_1^2}+\frac{1}{2\sigma_2^2} \\ \gamma &= \sqrt{\Lambda}/\sigma_2^2 \end{aligned}$$

The integral may be evaluated as (see Lindsey, Watson)

$$\int_{z=0}^{\infty}e^{-\alpha^2z^2}z^{L+2l}I_{L-1}(\gamma z)dz=\frac{l!\gamma^{L-1}}{2^L\alpha^{2(L+l)}}e^{\gamma^2/(4\alpha^2)}\sum_{k=0}^{l}\binom{l+L-1}{l-k}\frac{[\gamma^2/(4\alpha^2)]^k}{k!}$$

Thus

$$\begin{aligned} P\{Z_1>Z_2\} &= \sum_{l=0}^{L-1}\frac{1}{(2\sigma_1^2)^l l!\sigma_2^2\Lambda^{(L-1)/2}}\exp\{-\left(\frac{\Lambda}{2\sigma_2^2}\right)\}\frac{l!\gamma^{L-1}}{2^L\alpha^{2(L+l)}}e^{\gamma^2/(4\alpha^2)} \\ &\qquad \sum_{k=0}^{l}\binom{l+L-1}{l-k}\frac{[\gamma^2/(4\alpha^2)]^k}{k!} \\ &= \sum_{l=0}^{L-1}\frac{1}{(2\sigma_1^2)^l\sigma_2^2\Lambda^{(L-1)/2}}\exp\{-\left(\frac{\Lambda}{2\sigma_2^2}\right)\}\frac{l!\gamma^{L-1}}{2^L\alpha^{2(L+l)}}e^{\gamma^2/(4\alpha^2)} \end{aligned}$$

$$\sum_{k=0}^{l}\binom{l+L-1}{l-k}\frac{[\gamma^2/(4\alpha^2)]^k}{k!}$$

$$= \sum_{l=0}^{L-1}\frac{1}{(2\sigma_1^2)^l\sigma_2^2\Lambda^{(L-1)/2}}\exp\{\gamma^2/(4\alpha^2)-\frac{\Lambda}{2\sigma_2^2}\}\frac{\gamma^{L-1}}{2^L\alpha^{2(L+l)}}$$

$$\sum_{k=0}^{l}\binom{l+L-1}{l-k}\frac{[\gamma^2/(4\alpha^2)]^k}{k!}$$

$$\begin{aligned}
\alpha^2 &= \frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}\\
\gamma^2/(4\alpha^2) &= \frac{(\Lambda/\sigma_2^4)2\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\\
&= \frac{\Lambda\sigma_1^2}{2\sigma_2^2(\sigma_1^2+\sigma_2^2)}\\
\gamma^2/(4\alpha^2)-\frac{\Lambda}{2\sigma_2^2} &= \frac{(\Lambda/\sigma_2^4)2\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}-\frac{\Lambda}{2\sigma_2^2}\\
&= \frac{\Lambda}{2\sigma_2^2}\left[\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}-1\right]
\end{aligned}$$

$$= \frac{\Lambda}{2(\sigma_1^2+\sigma_2^2)}$$

$$\begin{aligned}
P\{Z_1>Z_2\} &= \exp\{-\frac{\Lambda}{2(\sigma_1^2+\sigma_2^2)}\}\sum_{l=0}^{L-1}\frac{1}{(2\sigma_1^2)^l\sigma_2^2\Lambda^{(L-1)/2}}\frac{\gamma^{L-1}}{2^L\alpha^{2(L+l)}}\\
&\quad \sum_{k=0}^{l}\binom{l+L-1}{l-k}\frac{[\gamma^2/(4\alpha^2)]^k}{k!}\\
&= e^{-\frac{\Lambda}{2(\sigma_1^2+\sigma_2^2)}}\left[\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right]^L\sum_{l=0}^{L-1}\left[\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right]^l\sum_{k=0}^{l}\binom{l+L-1}{l-k}\frac{[\gamma^2/(4\alpha^2)]^k}{k!}\\
&= e^{-\frac{\Lambda}{2(\sigma_1^2+\sigma_2^2)}}\left[\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right]^L\sum_{l=0}^{L-1}\left[\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right]^l\sum_{k=0}^{l}\binom{l+L-1}{l-k}\frac{1}{k!}\left[\frac{\Lambda\sigma_1^2}{2\sigma_2^2(\sigma_1^2+\sigma_2^2)}\right]^k
\end{aligned}$$

Performance as a function of L

Add curves for various L

Frequency Shift Keying (FSK)

Frequency shift keying communicates information by transmitting different frequencies. It can be demodulated noncoherently (by measuring the received energy at the different frequencies). It performance is worse than coherently demodulated signals but may be simpler.

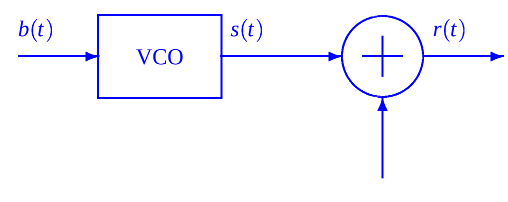


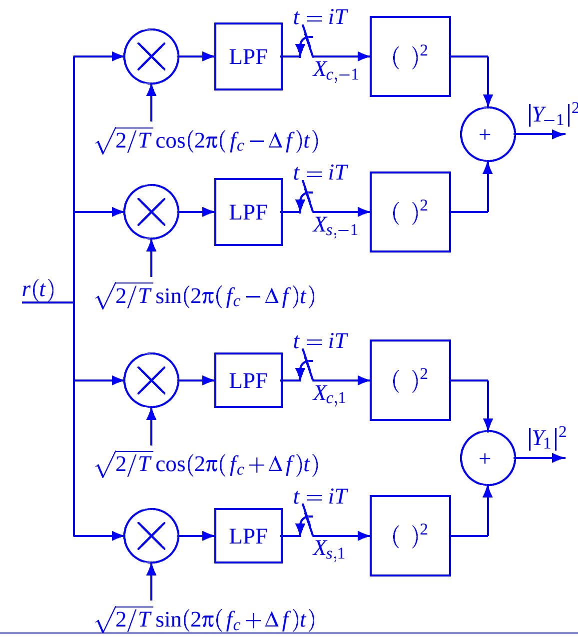
Figure 27: FSK Modulator

$$b(t) = \sum_{l=-\infty}^{\infty} b_l\, p_T(t-lT), \quad b_l \in \{+1,-1\}$$

$$s(t) = \sqrt{2P} \sum_{l=-\infty}^{\infty} \cos(2\pi(f_c + b(t)\Delta f)t + \theta) p_T(t - lT)$$

where Δf is half the difference between the two transmitted frequencies and θ is an unknown (to the receiver) phase. We let $f_0 = f - \Delta f$ and $f_1 = f + \Delta f$. When $b_i = +1$ then a signal at frequency f_1 is transmitted. When $b_i = -1$ then a signal at frequency f_0 is transmitted. The two frequencies f_0 and f_1 are separated far enough to make the two signals orthogonal. (Minimum shift keying has the minimum separation in order to make the signals orthogonal).

Noncoherent Demodulator



The receiver decides signal -1 was transmitted if $|Y_{-1}| > |Y_1|$ and otherwise decides signal 1. The random variables at the output of the low pass filters are

$$\begin{aligned}
X_{c,1}(iT) &= \sqrt{E}\delta(b_{i-1}, 1)\cos(\theta) + \eta_{c,1} \\
X_{s,1}(iT) &= \sqrt{E}\delta(b_{i-1}, 1)\sin(\theta) + \eta_{s,1} \\
X_{c,-1}(iT) &= \sqrt{E}\delta(b_{i-1}, -1)\cos(\theta) + \eta_{c,-1} \\
X_{s,-1}(iT) &= \sqrt{E}\delta(b_{i-1}, -1)\sin(\theta) + \eta_{s,-1}
\end{aligned}$$

where $\delta(a,b) = 1$ if $a = b$ and is zero otherwise. In the absence of noise ($\eta_{x,i} = 0$) it is easy to see that when $b_{i-1} = +1$ that $Y_1 = \sqrt{E}$ and $Y_{-1} = 0$. The error probability of binary FSK is

$$P_{e,b} = \frac{1}{2} e^{-E_b/2N_0}.$$

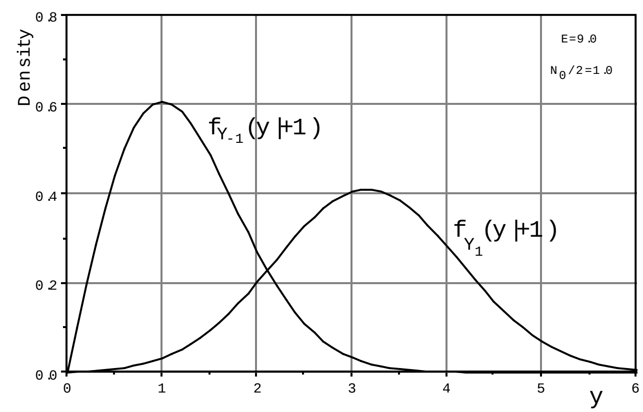


Figure 29: Output Densities For Noncoherent Receivers.

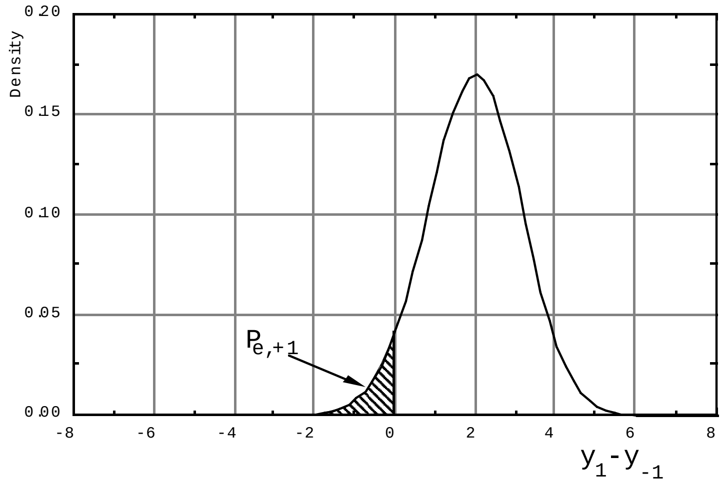


Figure 30: Density for $Y_1 - Y_{-1}$ given +1 Transmitted.

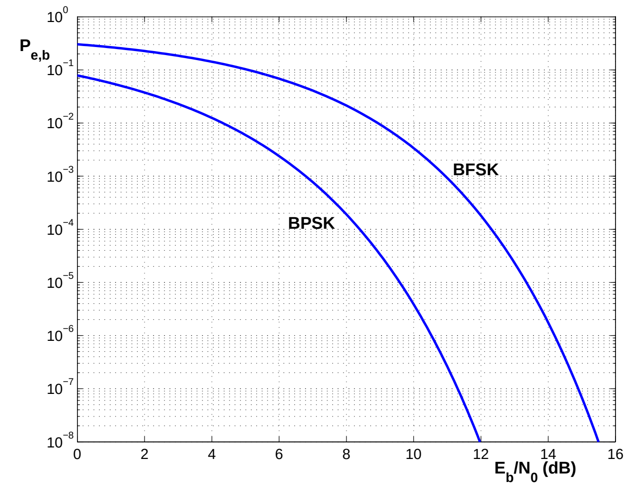
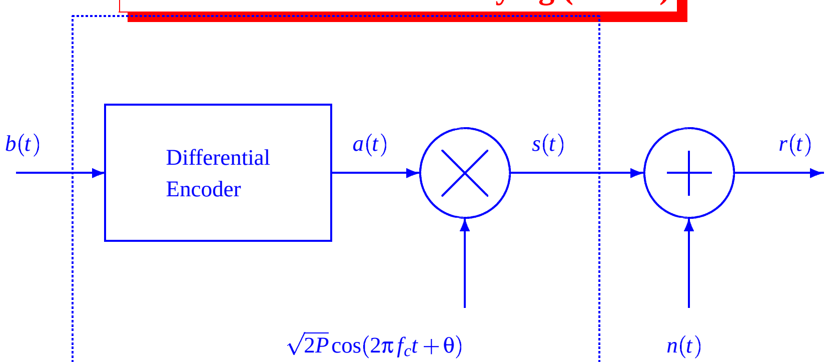


Figure 31: Error Probability of FSK with Noncoherent Detection.

Differential Phase Shift Keying (DPSK)



Modulator

$$b(t) = \sum_{l=-\infty}^{\infty} b_l \, p_T(t - lT), \quad b_l \in \{+1, -1\}.$$

$$a(t) = \sum_{l=-\infty}^{\infty} a_l \, p_T(t - lT), \quad a_l \in \{+1, -1\}.$$

Differential Encoder is such that

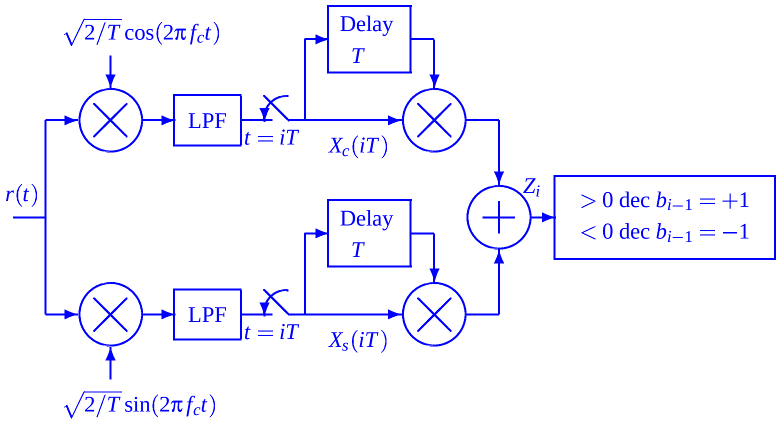
$$b_l = 1 \Rightarrow a_l = a_{l-1}$$
$$b_l = -1 \Rightarrow a_l = -a_{l-1}.$$

For example

l	…	−2	−1	0	1	2	3	…
b_l		−1	1	1	−1	1	−1	
a_l	−1	1	1	1	−1	−1	1	

$$s(t) = \sqrt{2P}a(t)\cos(2\pi f_c t + \theta).$$

DPSK Demodulator



$$X_c(iT) = \sqrt{E}a_{i-1}\cos\theta + \eta_{c,i}.$$

$$X_s(iT) = \sqrt{E}a_{i-1}\sin\theta + \eta_{s,i}.$$

The random variables $\eta_{c,i}$ and $\eta_{s,i}$ are independent identically distributed Gaussian random variables with mean 0 and variance $N_0/2$. Thus

$$Z_i = X_c(iT)X_c((i-1)T) + X_s(iT)X_s((i-1)T)$$
$$Z_i = \mathrm{Re}[W(iT)W^*((i-1)T)]$$

where $W(iT) = X_c(iT) - jX_s(iT)$. The error probability for DPSK is

$$P_{e,b} = \frac{1}{2}e^{-E/N_0}.$$

Thus differential phase shift keying is 3dB better than FSK with noncoherent detection. However, errors tend to occur in pairs.

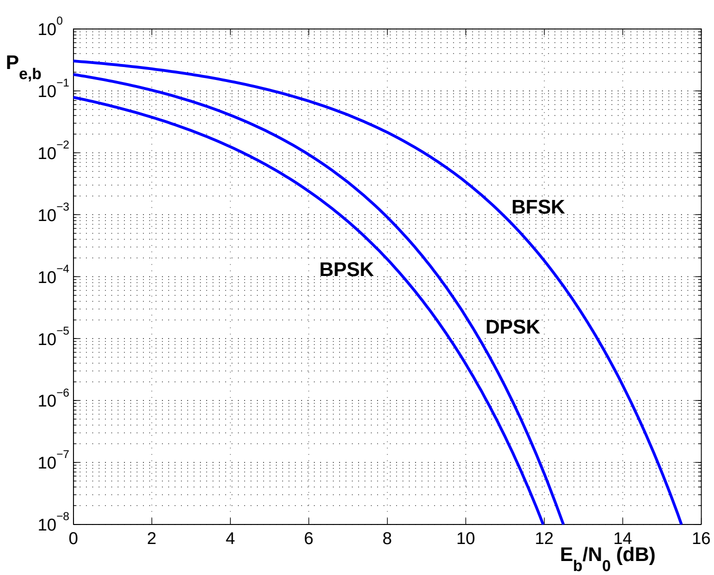


Figure 32: Error Probability for Differential Phase Shift Keying

Error Probability

To derive the above expression for DPSK consider the low pas filter with impulse response $h(t) = p_T(t)$. The output of the lowpass filters can be expressed as

$$
\begin{aligned}
X_c(t) &= \int_{-\infty}^{\infty} 2\cos\omega_c\tau h(t-\tau)r(\tau)d\tau \\
X_c(iT) &= \int_{-\infty}^{\infty} 2\cos\omega_c\tau p_T(iT-\tau)r(\tau)d\tau \\
&= \int_{(i-1)T}^{iT} 2\cos\omega_c\tau\left[\sum_{l=-\infty}^{\infty}\sqrt{2P}a_l\cos(\omega_c\tau+\theta)p_T(\tau-lT)+n(\tau)\right]d\tau \\
&= \int_{(i-1)T}^{iT}\sqrt{2P}2a_{i-1}\cos\omega_c\tau\cos(\omega_c\tau+\theta)d\tau + \eta_{c,i}
\end{aligned}
$$

$n_{c,i}$ is Gaussian random variable, mean 0 variance $N_0/2$. Assuming $\omega_c T = 2\pi n$

$$X_c(iT) = \sqrt{2P}a_{i-1}T\cos\theta + \eta_{c,i}$$

Similarly

$$X_s(iT) = \sqrt{2P}a_{i-1}T\sin\theta + n_{c,i}$$

Thus

$$Z_i = X_c(iT)X_c((i-1)T) + X_s(iT)X_s((i-1)T)$$

Note that if we write $W(iT) = X_c(iT) - jX_s(iT)$ that $Z_i = \mathrm{Re}[W(iT)W^*((i-1)T)]$. It is clear that this represents the phase difference between two consecutive symbols.

Let

$$\begin{aligned} U_1 &= \frac{X_c(iT) + X_c((i-1)T)}{2} \\ U_2 &= \frac{X_s(iT) + X_s((i-1)T)}{2} \\ U_3 &= \frac{X_c(iT) - X_c((i-1)T)}{2} \\ U_4 &= \frac{X_s(iT) - X_s((i-1)T)}{2} \end{aligned}$$

$$Z_i = U_1^2 + U_2^2 - (U_3^2 + U_4^2)$$

Assume $b_{i-1} = +1$ so that $a_{i-1} = a_{i-2}$ then

$$\begin{aligned} P_{e,0} &= P\{Z < 0 |\, a_{i-1} = a_{i-2}\} \\ &= P\left\{U_1^2 + U_2^2 \leq U_3^2 + U_4^2\right\} \\ U_1 &\sim N(\mu_1, \sigma^2) \end{aligned}$$

$$\begin{aligned} U_2 &\sim N(\mu_2, \sigma^2) \\ \mu_1 &= \frac{1}{2}\sqrt{2P}(a_{i-1}T\cos\theta + a_{i-2}T\cos\theta) \\ &= \frac{1}{2}\sqrt{2P}(a_{i-1} + a_{i-2})T\cos\theta \\ \mu_2 &= \frac{1}{2}\sqrt{2P}(a_{i-1} + a_{i-2})T\sin\theta \\ \sigma^2 &= \frac{1}{4}[N_0T + N_0T] \\ &= \frac{1}{2}N_0T \\ U_3 &\sim N(\mu_3, \sigma^2) \quad U_4 \sim N(\mu_4, \sigma^2) \\ \mu_3 &= 0, \qquad \mu_4 = 0, \end{aligned}$$

$$\begin{aligned} E[U_1U_2] &= E\left[\frac{X_c(iT) + X_c((i-1)T)}{2}\right]\left[\frac{X_s(iT) + X_s((i-1)T)}{2}\right] \\ &= \frac{1}{4}E\left[X_c(iT)X_s(iT) + X_c(iT)X_s((i-1)T)\right. \\ &\quad \left. + X_c((i-1)T)X_s(iT) + X_c((i-1)T)X_s((i-1)T)\right] \end{aligned}$$

$$\begin{aligned} E\left[X_c(iT)X_s(jT)\right] &= E[X_c(iT)]E[X_s(jT)] \text{ since independent} \\ E[U_1U_2] &= \left(\frac{E\left[X_c(iT)\right] + E\left[X_c((i-1)T)\right]}{2}\right)\left(\frac{E\left[X_s(iT)\right] + E\left[X_s((i-1)T)\right]}{2}\right) \\ &= E[U_1]E[U_2] \Rightarrow U_1, U_2 \text{ independent} \end{aligned}$$

Similarly (U_1, U_3) independent (U_2, U_3) independent (U_3, U_4) independent (U_1, U_4) independent $U_2, U_4)$ independent

Thus $U_1^2 + U_2^2$ is independent of $U_3^2 + U_4^2$. From the results derived for noncoherent FSK it is easy to show that

$$P\{U_1^2 + U_2^2 \leq U_3^2 + U_4^2\} = \frac{1}{2}e^{-E/N_0}.$$

Thus differential phase shift keying is 3dB better than FSK with noncoherent detection. However, errors tend to occur in pairs.