Lecture Notes 6: Basic Modulation Schemes

In this lecture we examine a number of different simple modulation schemes. We examine the implementation of the optimum receiver, the error probability and the bandwidth occupancy. We would like the simplest possible receiver, with the lowest error probability and smallest bandwidth for a given data rate.

VI-1

Binary Phase Shift Keying (BPSK)

The first modulation considered is binary phase shift keying. In this scheme during every bit duration, denoted by $T$, one of two phases of the carrier is transmitted. These two phases are 180 degrees apart. This makes these two waveforms antipodal. Any binary modulation where the two signals are antipodal gives the minimum error probability (for fixed energy) over any other set of binary signals. The error probability can only be made smaller (for fixed energy per bit) by allowing more than two waveforms for transmitting information.

VI-2

BPSK Modulator

![Figure 33: Modulator for BPSK](image)

To mathematically described the transmitted signal we define a pulse function $p_T(t)$ as

$$p_T(t) = \begin{cases} 1, & 0 \leq t \leq T \\ 0, & \text{otherwise.} \end{cases}$$

Let $b(t)$ denote the data waveform consisting of an infinite sequence of pulses of duration $T$

$$b(t) = \sum_{l=-\infty}^{\infty} b_l p_T(t-lT)$$

and height $\pm 1$.

$$b(t) = \sum_{l=-\infty}^{\infty} b_l p_T(t-lT), \quad b_l \in \{+1,-1\}.$$
Figure 34: Data and Phase waveforms for BPSK

The optimum receiver for BPSK in the presence of additive white Gaussian noise is shown in Figure VI-3. The low pass filter (LPF) is a filter “matched” to the baseband signal being transmitted. For BPSK this is just a rectangular pulse of duration $T$. The impulse response is

$$h(t) = p_T(t)$$

The output of the low pass filter is

$$X(t) = \int_{-\infty}^{\infty} \sqrt{2/T} \cos(2\pi f_c \tau) b(t - \tau) r(\tau) d\tau,$$

$\eta_i$ is Gaussian random variable, mean 0 variance $N_0/2$. Assuming $2\pi f_c T = 2\pi n$ for some integer $n$ (or that $f_c T > 1$)

$$X(iT) = \sqrt{P} b_{i-1} + \eta_i = \sqrt{E} b_{i-1} + \eta_i.$$

The sampled version of the output is given by

$$X(iT) = \int_{-\infty}^{\infty} \sqrt{2/T} \cos(2\pi f_c \tau) p_T(iT - \tau) r(\tau) d\tau$$

$$= \int_{(i-1)T}^{iT} \sqrt{2/T} \cos(2\pi f_c \tau) \left[ \sqrt{2T} h(\tau) \cos(2\pi f_c \tau) ight] d\tau$$

$$+ n(\tau) d\tau$$

$$= \int_{(i-1)T}^{iT} 2\sqrt{P/T} b_{i-1} \cos(2\pi f_c \tau) \cos(2\pi f_c \tau) d\tau + \eta_i.$$

Figure 35: Demodulator for BPSK

$$r(t) \rightarrow \text{LPF} \rightarrow t = iT \rightarrow \begin{cases} > 0 \text{ dec } b_{i-1} = +1 \\ < 0 \text{ dec } b_{i-1} = -1 \end{cases}$$

Figure 36: Probability Density of Decision Statistic for Binary Phase Shift Keying
Bit Error Probability of BPSK

\[ P_{e,b} = Q\left(\sqrt{\frac{2E}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \]

where

\[ Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \]

For binary signals this is the smallest bit error probability, i.e. BPSK are optimal signals and the receiver shown above is optimum (in additive white Gaussian noise). For binary signals the energy transmitted per information bit \( E_b \) is equal to the energy per signal \( E \). For \( P_{e,b} = 10^{-5} \) we need a bit-energy, \( E_b \) to noise density \( N_0 \) ratio of \( E_b/N_0 = 9.6 \text{dB} \). Note: \( Q(x) \) is a decreasing function which is \( 1/2 \) at \( x = 0 \). There are efficient algorithms (based on Taylor series expansions) to calculate \( Q(x) \). Since \( Q(x) \leq e^{-x^2/2}/2 \) the error probability can be upper bounded by

\[ P_{e,b} \leq \frac{1}{2} e^{(E_b/N_0)} \]

which decreases exponentially with signal-to-noise ratio.

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Bandwidth of BPSK

The power spectral density is a measure of the distribution of power with respect to frequency. The power spectral density for BPSK has the form

\[ S(f) = \frac{PT}{2} \left[ \sin^2((f - f_c)T) + \sin^2((f + f_c)T) \right] \]

where

\[ \sin(x) = \frac{\sin(\pi x)}{\pi x} \]

Notice that

\[ \int_{-\infty}^{\infty} S(f) df = P. \]

The power spectrum has zeros or nulls at \( f = f_c = i/T \) except for \( i = 0 \); that is there is a null at \( f = f_c = \pm 1/T \) called the first null; a null at \( f = f_c = \pm 2/T \) called the second null; etc. The bandwidth between the first nulls is called the null-to-null bandwidth. For BPSK the null-to-null bandwidth is \( 2/T \). Notice that the spectrum falls off as \( (f - f_c)^2 \) as \( f \) moves away from \( f_c \). (The spectrum of MSK falls off as the fourth power, versus the second power for BPSK).

It is possible to reduce the bandwidth of a BPSK signal by filtering. If the filtering is done properly the (absolute) bandwidth of the signal can be reduced to \( 1/T \) without causing any intersymbol interference; that is all the power is concentrated in the frequency range

\[ -1/(2T) \leq |f - f_c| \leq 1/(2T). \]

The drawbacks are that the signal loses its constant envelope property (useful for nonlinear amplifiers) and the sensitivity to timing errors is greatly increased. The timing sensitivity problem can be greatly alleviated by filtering to a slightly larger bandwidth \( -\alpha/(2T) \leq |f - f_c| \leq (1 + \alpha)/(2T) \).
Example

Given:
- Noise power spectral density of $N_0/2 = -180$ dBm/Hz = $10^{-21}$ Watts/Hz.
- $P_T = 3 \times 10^{13}$ Watts
- Desired $P_e = 10^{-7}$.

Find: The data rate that can be used and the bandwidth that is needed.

Solution: Need $Q(\sqrt{2E_b/N_0}) = 10^{-7}$ or $E_b/N_0 = 11.3$dB or $E_b/N_0 = 13.52$. But $E_b/N_0 = P_T/N_0 = 13.52$. Thus the data bit must be at least $T = 9.0 \times 10^{-8}$ seconds long, i.e. the data rate $1/T$ must be less than 11 Mbits/second. Clearly we also need a (null-to-null) bandwidth of 22 MHz.

An alternative view of BPSK is that of two antipodal signals; that is

\[ s_0(t) = \sqrt{E} \psi(t), \quad 0 \leq t \leq T \]

and

\[ s_1(t) = -\sqrt{E} \psi(t), \quad 0 \leq t \leq T \]

where $\psi(t) = \sqrt{2/T} \cos(2\pi f_c t)$, $0 \leq t \leq T$ is a unit energy waveform. The above describes the signals transmitted only during the interval $[0, T]$. Obviously this is repeated for other
intervals. The receiver correlates with $\psi(t)$ over the interval $[0, T]$ and compares with a threshold (usually 0) to make a decision. The correlation receiver is shown below.

\[
\begin{array}{c}
\psi(t) \\
\downarrow \\
\times \\
\downarrow \\
r(t) \\
\hline
\end{array}
\]

\[
\begin{array}{c}
<T \text{dec } s_0 \\
<T \text{dec } s_1 \\
\hline
\end{array}
\]

This is called the “Correlation Receiver.” Note that synchronization to the symbol timing and oscillator phase are required.

**Effect of Filtering and Nonlinear Amplification on a BPSK waveform**

In this section we illustrate one main drawback to BPSK. The fact that the signal amplitude has discontinuities causes the spectrum to have fairly large sidelobes. For a system that has a constraint on the bandwidth this can be a problem. A possible solution is to filter the signal. A bandpass filter centered at the carrier frequency which removes the sidbands can be inserted after mixing to the carrier frequency. Alternatively we can filter the data signal at baseband before mixing to the carrier frequency.

Below we simulate this type of system to illustrate the effect of filtering and nonlinear amplification. The data waveform $b(t)$ is mixed onto a carrier. This modulated waveform is denoted by

\[
s_1(t) = \sqrt{2P} \cos(2\pi f_c t)
\]

The signal $s_1(t)$ is filtered by a fourth order bandpass Butterworth filter with passband from $f_c - 4Rb$ to $f_c + 4Rb$. The filtered signal is denoted by $s_2(t)$. The signal $s_2(t)$ is then amplified.

The input-output characteristics of the amplifier are

\[
s_3(t) = 100 \tanh(2s_1(t))
\]

This amplifier is fairly close to a hard limiter in which every input greater than zero is mapped to 100 and every input less than zero is mapped to -100.

**Simulation Parameters**
- Sampling Frequency= 50MHz
- Sampling Time =20nseconds
- Center Frequency= 12.5MHz
- Data Rate=390.125kbps
- Simulation Time= 1.31072 m s
**Quaternary Phase Shift Keying (QPSK)**

The next modulation technique we consider is QPSK. In this modulation technique one of four phases of the carrier is transmitted in a symbol duration denoted by $T_s$. Since one of four waveforms is transmitted there are two bits of information transmitted during each symbol duration. An alternative way of describing QPSK is that of two carriers offset in phase by 90 degrees. Each of these carriers is then modulated using BPSK. These two carriers are called the inphase and quadrature carriers. Because the carriers are 90 degrees offset, at the output of the correlation receiver they do not interfere with each other (assuming perfect phase synchronization). The advantage of QPSK over BPSK is that the data rate is twice as high for the same bandwidth. Alternatively single-sideband BPSK would have the same rate in bits per second per hertz but would have a more difficult job of recovering the carrier frequency and phase.
\[ b_i(t) = \sum_{l=-\infty}^{\infty} b_{i,l} p_{T_c}(t - lT_s), \quad b_{i,l} \in \{+1, -1\} \]

\[ b_i(t) = \sum_{l=-\infty}^{\infty} b_{i,l} p_{T_c}(t - lT_s), \quad b_{i,l} \in \{+1, -1\} \]

\[ s(t) = \sqrt{P} [b_i(t) \cos(2\pi f_c t) - b_i(t) \sin(2\pi f_c t)] \]
\[ = \sqrt{2P} \cos(2\pi f_c t + \phi(t)) \]

The transmitted power is still \( P \). The symbol duration is \( T_s \) seconds. The data rate is \( R_b = 2/T_s \) bits/seconds.

The phase \( \phi(t) \) of the transmitted signal is related to the data waveform as follows.

\[ \phi(t) = \sum_{l=-\infty}^{\infty} \phi_l p_{T_c}(t - lT_s), \quad \phi_l \in \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\} \]

The relation between \( \phi_l \) and \( b_{i,l}, b_{i,l} \) is shown in the following table

<table>
<thead>
<tr>
<th>( b_{i,l} )</th>
<th>( b_{i,l} )</th>
<th>( \phi_l )</th>
</tr>
</thead>
<tbody>
<tr>
<td>+1</td>
<td>+1</td>
<td>( \pi/4 )</td>
</tr>
<tr>
<td>-1</td>
<td>+1</td>
<td>( 3\pi/4 )</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>( 5\pi/4 )</td>
</tr>
<tr>
<td>+1</td>
<td>-1</td>
<td>( 7\pi/4 )</td>
</tr>
</tbody>
</table>

The constellation of QPSK is shown below. The phase of the overall carrier can be on of four values. Transitions between any of the four values may occur at any symbol transition. Because of this, it is possible that the transition is to the 180 degree opposite phase. When this happens the amplitude of the signal goes through zero. In theory this is an instantaneous transition. In practice, when the signal has been filtered to remove out-of-band components this transition is slowed down. During this transition the amplitude of the carrier goes through zero. This can be undesirable for various reasons. One reason is that nonlinear amplifiers with a non constant envelope signal will regenerate the out-of-band spectral components. Another reason is that at the receiver, certain synchronization circuits need constant envelope to maintain their tracking capability.

Figure 42: Timing and Phase of QPSK
The bandwidth of QPSK is given by

\[
S(f) = \frac{P T_s}{2} \left( \text{sinc}^2 \left( \frac{(f - f_c) T_s}{2} \right) + \text{sinc}^2 \left( \frac{(f + f_c) T_s}{2} \right) \right)
\]

\[
= P T_b \left[ \text{sinc}^2 \left( 2(f - f_c) T_b \right) + \text{sinc}^2 \left( 2(f + f_c) T_b \right) \right]
\]

since \( T_s = T_b / 2 \). Thus while the spectrum is compressed by a factor of 2 relative to BPSK with the same bit rate, the center lobe is also 3dB higher, that is the peak power density is higher for QPSK than BPSK. The null-to-null bandwidth is \( 2/T_s = R_b \).
Assuming $2\pi f_c T_s = 2\pi n$ or $2\pi f_c T_s \gg 1$

$$X_c(iT_s) = \sqrt{PT_s/2} b_{c,i} + \eta_{c,i} = \sqrt{E_b} b_{c,i} + \eta_{c,i}$$

$$X_s(iT_s) = \sqrt{PT_s/2} b_{s,i} + \eta_{s,i} = \sqrt{E_b} b_{s,i} + \eta_{s,i}$$

where $E_b = PT_s/2$ is the energy per transmitted bit. Also $\eta_{c,i}$ and $\eta_{s,i}$ are Gaussian random variables, with mean 0 and variance $N_0/2$.

Bit Error Probability of QPSK

$$P_{e,b} = Q(\sqrt{\frac{2E_b}{N_0}})$$

The probability that a symbol error is made is

$$P_{e,s} = 1 - (1 - P_{e,b})^2 = 2P_{e,b} - P_{e,b}^2$$

Thus for the same data rate, transmitted power, and bit error rate (probability of error), QPSK has half the (null-to-null) bandwidth of BPSK.

Example

Given:
- Noise power spectral density of $N_0/2 = -110$ dBm/Hz = 10^{-14} Watts/Hz.
- $P_r = 3 \times 10^6$ Watts
- Desired $P_e = 10^{-7}$.

Find: The data rate that can be used and the bandwidth that is needed for QPSK.

Solution: Need $Q(\sqrt{2E_b/N_0}) = 10^{-7}$ or $E_b/N_0 = 11,3$dB or $E_b/N_0 = 13,52$. But

$$E_b/N_0 = \frac{\frac{P_r(T_s)}{N_0}}{N_0} = \frac{P_r T_s}{N_0} = 13,52$$

since $T_s = 2T$. Thus the data bit must be at least $T = 9,0 \times 10^{-8}$ seconds long, i.e. the data rate $1/T$ must be less than 11 Mbits/second. Clearly we also need a (null-to-null) bandwidth of 11 MHz.
Offset Quaternary Phase Shift Keying (OQPSK)

The disadvantages of QPSK can be fixed by offsetting one of the data streams by a fraction (usually 1/2) of a symbol duration. By doing this we only allow one data bit to change at a time. When this is done the possible phase transitions are ±90°. In this way the transitions through the origin are eliminated. Offset QPSK then gives the same performance as QPSK but will have less distortion when there is filtering and nonlinearities.

\[ s(t) = \sqrt{P}[b_i(t - T_s/2) \cos(2\pi f_c t) - b_s(t) \sin(2\pi f_c t)] \]
\[ s(t) = \sqrt{2P}\cos(2\pi f_c t + \phi(t)) \]

The transmitted power is still \( P \). The symbols duration is \( T_s \) seconds. The data rate is \( R_b = 2/T_s \) bits seconds. The bandwidth (null-to-null) is \( 2/T_c = R_b \). This modification of QPSK removes the possibility of both data bits changing simultaneously. However, one of the data bits may change every \( T_s/2 \) seconds but 180 degree changes are not allowed. The bandwidth of OQPKS is the same as QPSK. OQPSK has advantage over QPSK when passed through nonlinearities (such as in a satellite) in that the out of band interference generated by first bandlimiting and then hard limiting is less with OQPSK than QPSK.
Assuming $2\pi f_c T_s = 2\pi n$ or $2\pi f_c T_s \gg 1$

$$X(i T_s - T_s/2) = \sqrt{P T_s/2} b_{c,i} + \eta_{c,i} = \sqrt{E_b} b_{c,i} + \eta_{c,i}$$
$$X(i T_s) = \sqrt{P T_s/2} b_{s,i} + \eta_{s,i} = \sqrt{E_b} b_{s,i} + \eta_{s,i}$$

where $E_b = P T_s/2$ is the energy per transmitted bit. Also $\eta_{c,i}$ and $\eta_{s,i}$ are Gaussian random variables, with mean 0 variance $N_0/2$.

Bit Error Probability of OQPSK

$$P_{e,b} = Q(\sqrt{\frac{E_b}{N_0}})$$

The probability that a symbol error is made is

$$P_{e,s} = 1 - (1 - P_{e,b})^2 = 2P_{e,b} - P_{e,b}^2$$

This is the same as QPSK.

Minimum Shift Keying (MSK)

Minimum shift keying can be viewed in several different ways and has a number of significant advantages over the previously considered modulation schemes. MSK can be thought of as a variant of OQPSK where the data pulse waveforms are shaped to allow smooth transition between phases. It can also be thought of a form of frequency shift keying where the two frequencies are separated by the minimum amount to maintain orthogonality and have continuous phase when switching from one frequency to another (hence the name minimum shift keying). The advantages of MSK include a better spectral efficiency in most cases. In fact the spectrum of MSK falls off at a faster rate than BPSK, QPSK and OQPSK. In addition there is an easier implementation than OQPSK (called serial MSK) that avoids the problem of having a precisely controlled time offset between the two data streams. An additional advantage is that MSK can be demodulator noncoherently as well as coherently. So for applications requiring a low cost receiver MSK may be a good choice.
In the above table, because of the delay of the bit stream corresponding to the cosine branch, only one bit is allowed to change at a time. During each time interval of duration $T_s/2$ during which the data bits remain constant there is a phase shift of $\pm \pi/2$. Because the phase changes linearly with time MSK can also be viewed as frequency shift keying. The two different frequencies are $f_c + \frac{\pi f_b}{T_s}$ and $f_c - \frac{\pi f_b}{T_s}$. The change in frequency is $\Delta f = \frac{\pi f_b}{T_s}$. The change in frequency depends on whether $f_c + \frac{\pi f_b}{T_s}$ or $f_c - \frac{\pi f_b}{T_s}$ is used. The transmitted power is still $P$. The symbols duration is $T_s$. The data rate is $R_b = \frac{2}{T_s}$ bits per second. The signal has constant envelope which is useful for nonlinear amplifiers. The bandwidth is different because of the pulse shaping waveforms.
The spectrum of MSK is given by

\[
S(f) = \frac{8PT_b}{\pi^2} \left[ \frac{\cos^2(2\pi f T_b)}{[1 - (4T_b(f - f_c))^2]^2} + \frac{\cos^2(2\pi f T_b)}{[1 - (4T_b(f + f_c))^2]^2} \right]
\]

The nulls in the spectrum are at \((f - f_c)T_b = 0.75, 1.25, 1.75, \ldots\). Because we force the signal to be continuous in phase MSK has significantly faster decay of the power spectrum as the frequency from the carrier becomes larger. MSK decays as \(1/f^4\) while QPSK, OQPSK, and BPSK decay as \(1/f^2\) as the frequency differs more and more from the center frequency.
Figure 56: Spectrum of MSK

Figure 57: Spectrum of MSK

Figure 58: Spectrum of MSK

Figure 59: Coherent Demodulator for MSK
Assuming $2\pi f_c T_s = 2\pi n$ or $2\pi f_c T_s \gg 1$

\begin{align*}
X_i(t_i + T_i/2) &= \sqrt{P T_s / 2} b_{c,i} + \eta_{c,i} = \sqrt{E_b} b_{c,i} + \eta_{c,i} \\
X_0(t_i) &= \sqrt{P T_s / 2} b_{s,i} + \eta_{s,i} = \sqrt{E_b} b_{s,i} + \eta_{s,i}
\end{align*}

where $E_b = P T_s / 2$ is the energy per transmitted bit. Also $\eta_{c,i}$ and $\eta_{s,i}$ are Gaussian random variables, with mean 0 variance $N_0/2$.

Bit Error Probability of MSK with Coherent Demodulation

Since the signals are still antipodal

\[ P_{e_b} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \]

The probability that a symbol error is made is

\[ P_{e_s} = 1 - (1 - P_{e_b})^2 = 2P_{e_b} - P_{e_b}^2 \]

Figure 60: Waveform for Minimum Shift Keying

Figure 61: Phase Waveform for Minimum Shift Keying
Noncoherent Demodulation of MSK

Because MSK can be viewed as a form of Frequency Shift Keying it can also be demodulated noncoherently. For the same sequence of data bits the frequency is \( f_c = 1/2T_s \) if 
\[ b_c(t - T_s/2) = b_s(t) \] and is \( f_c = 1/2T_s \) if \( b_c(t - T_s/2) \neq b_s(t) \).

Consider determining \( b_{s,i-1} \) at time \((i-1)T_s\). Assume we have already determined \( b_{s,i-2} \) at time \((i-1)T_s\). If we estimate which of two frequencies is sent over the interval 
\[ [(i-1)T_s, (i-1)T_s] \] the decision rule is to decide that \( b_{s,i-1} = b_{s,i-2} \) if the frequency detected is \( f_c = 1/(2T_s) \) and to decide that \( b_{s,i-1} = -b_{s,i-2} \) if the frequency detected is \( f_c = 1/(2T_s) \).

Consider determining \( b_{c,i-1} \) at time \(iT_s\). Assume we have already determined \( b_{c,i-1} \) at time \((i-1)T_s\). If we estimate which of two frequencies is sent over the interval 
\[ [(i-1)T_s, iT_s] \] the decision rule is to decide that \( b_{c,i-1} = b_{s,i-1} \) if the frequency detected is \( f_c = 1/(2T_s) \) and to decide that \( b_{c,i-1} = -b_{s,i-1} \) if the frequency detected is \( f_c = 1/(2T_s) \).

The method to detect which of the two frequencies is transmitted is identical to that of Frequency Shift Keying which will be considered later.

For the example phase waveform shown previously we have that

<table>
<thead>
<tr>
<th>Time Interval</th>
<th>([0, T_s/2])</th>
<th>([T_s/2, T_s])</th>
<th>([T_s, 3T_s/2])</th>
<th>([3T_s/2, 2T_s])</th>
<th>([2T_s, 5T_s/2])</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>+</td>
<td>+</td>
<td>−</td>
<td>+</td>
<td>−</td>
</tr>
<tr>
<td>Previous Data</td>
<td>(b_{c,i-1} = +1)</td>
<td>(b_{s,i} = -1)</td>
<td>(b_{s,i-1} = 1)</td>
<td>(b_{c,i-1} = +1)</td>
<td>(b_{c,i} = -1)</td>
</tr>
<tr>
<td>Detected Data</td>
<td>(b_{c,i} = -1)</td>
<td>(b_{s,i-1} = +1)</td>
<td>(b_{c,i-1} = +1)</td>
<td>(b_{c,1} = -1)</td>
<td>(b_{c,2} = -1)</td>
</tr>
</tbody>
</table>

So detecting the frequency can also be used to detect the data.
Serial Modulation and Demodulation

The implementation of MSK as parallel branches suffers from significant sensitivity to precise timing of the data (exact shift by $T$ for the inphase component) and the exact balance between the inphase and quadriphase carrier signals. An alternative implementation of MSK that is less complex and does not have these drawbacks is known as serial MSK. Serial MSK does have an additional restriction that $f_c = (2n + 1)/4T$ which may be important when $f_c$ is about the same as $1/T$ but for $f_c \gg 1/T$ it is not important. The block diagram for serial MSK modulator and demodulator is shown below.

![Block diagram of serial MSK modulator and demodulator](image)

The filter $G(f)$ is given by

$$G(f) = T \text{sinc}[(f - f_1)T]e^{-j\pi f_1 T} + T \text{sinc}[(f + f_1)T]e^{-j\pi f_1 T}$$

Demodulator

The low pass filter (LPF) removes double frequency components. Serial MSK is also be viewed as a filtered form of BPSK where the BPSK signal center frequency is $f_1$ but the filter is not symmetric with respect to $f_1$. The receiver is a filter matched to the transmitted signal (and hence optimal). The output is then mixed down to baseband where it is filtered (to remove the double frequency terms) and sampled.

$$g(t) = 2 \sin(2\pi f_1 t) p_T(t)$$

where $f_1 = f_c - \frac{1}{2T}$ and $f_2 = f_c + \frac{1}{2T}$. (For serial MSK we require $f_c = (2n + 1)/4T$ for some integer $n$. Otherwise the implementation does not give constant envelope).

$$H(f) = \frac{4T}{\pi} \frac{\cos[2\pi(f - f_1)T - 0.25]}{1 - 16[(f - f_1)T - 0.25]^2} e^{j2\pi f_1 T}$$
Continuous Phase Modulation

MSK is a special case of a more general form of modulation known as continuous phase modulation where the phase is continuous. The general form of CPM is given by

\[ s(t) = \sqrt{2P} \cos(2\pi f_c t + \phi(t)) \]

where the phase waveform has the form

\[ \phi(t) = 2\pi h \int_0^t \sum_{k=0}^{\infty} b_k g(t - kT) dt + \phi_0 \quad kT \leq t \leq (k+1)T \]

\[ = 2\pi h \sum_{k=0}^{\infty} b_k q(t - kT) + \phi_0 \quad kT \leq t \leq (k+1)T \]

The function \( g(\cdot) \) is the (instantaneous) frequency function, \( h \) is called the modulation index and \( b_i \) is the data. The function \( q(t) = \int_0^t g(\tau) d\tau \) is the phase waveform. The function \( g(t) = \frac{dg(t)}{dt} \) is the frequency waveform.

For example if CPM has \( h = 1/2 \) and

\[ q(t) = \begin{cases} 
0, & t < 0 \\
1/2, & 0 \leq t < T \\
1/2, & t > T,
\end{cases} \]

then the modulation is the same as MSK. Continuous Phase Modulation Techniques have constant envelope which make them useful for systems involving nonlinear amplifiers which also must have very narrow spectral widths.

Example

Given:
- Noise power spectral density of \( N_0/2 = -110 \text{ dBm/Hz} = 10^{-14} \text{ Watts/Hz}. \)
- \( P_r = 3 \times 10^{-6} \) Watts
- Desired \( P_e = 10^{-7}. \)
- Bandwidth available=26MHz (at the 902-928MHz band). The peak power outside must be 20dB below the peak power inside the band.

Find: The data rate that can be used for MSK.

Solution: Need \( Q(\sqrt{2E_b/N_0}) = 10^{-7} \) or \( E_b/N_0 = 11.3 \text{dB} \) or \( E_b/N_0 = 13.52. \) But \( E_b/N_0 = P_r T / N_0 = 13.52. \) Thus the data bit must be at least \( T = 9.0 \times 10^{-8} \) seconds long, i.e., the data rate \( 1/T \) must be less than 11 Mbits/second.

Gaussian Minimum Shift Keying

Gaussian minimum shift keying is a special case of continuous phase modulation discussed in the previous section. For GMSK the pulse waveforms are given by

\[ g(t) = Q\left(\frac{t-T}{\sigma}\right) - Q\left(\frac{t}{\sigma}\right) \]
Figure 68: Waveform for Gaussian Minimum Shift Keying (BT=0.3)

Figure 69: Waveform for Gaussian Minimum Shift Keying (BT=0.3)

Figure 70: Waveform for Gaussian Minimum Shift Keying (BT=0.3)

Figure 71: Waveform for Gaussian Minimum Shift Keying (BT=0.3)
As mentioned earlier the effect of filtering and nonlinearly amplifying a QPSK waveform causes distortion when the signal amplitude fluctuates significantly. Another modulation scheme that has less fluctuation that QPSK is $\pi/4$ QPSK. In this modulation scheme every other symbol is sent using a rotated (by 45 degrees) constellation. Thus the transitions from one phase to the next are still instantaneous (without any filtering) but the signal never makes a transition through the origin. Only $\pm 45$ and $\pm 135$ degree transitions are possible. This is shown in the constellation below where a little bit of filtering was done.
**Lecture 6b: Other Modulation Techniques**

**Orthogonal Signals**

A set of signals \( \{ \psi_i(t) : 0 \leq t \leq T, 0 \leq i \leq M-1 \} \) are said to be orthogonal (over the interval \([0, T]\)) if

\[
\int_0^T \psi_i(t)\psi_j(t)\,dt = 0, \quad i \neq j.
\]

In most cases the signals will have the same energy and it is convenient to normalize the signals to unit energy. A set of signals \( \{ \psi_i(t) : 0 \leq t \leq T, 1 \leq i \leq M \} \) are said to be orthonormal (over the interval \([0, T]\)) if

\[
\int_0^T \psi_i(t)\psi_j(t)\,dt = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}
\]

Many signal sets can be described as linear combinations of orthonormal signal sets as we will show later. Below we describe a number of different orthonormal signal sets. The signal sets will all be described at some intermediate frequency \( f_0 \) but are typically modulated up to the carrier frequency \( f_c \).
The symbol error probability can be upper bounded as

\[ P_{e,s} \leq \begin{cases} 1, & \frac{E_b}{N_0} \leq \ln M \\ \exp \left\{ -\left( \sqrt{\frac{E_b}{N_0}} - \sqrt{\ln M} \right)^2 \right\}, & \ln M \leq \frac{E_b}{N_0} \leq 4\ln M \\ \exp \left\{ -\left( \frac{E_b}{4N_0} - \ln M \right) \right\}, & \frac{E_b}{N_0} \geq 4\ln M. \end{cases} \]

Normally a communication engineer is more concerned with the energy transmitted per bit rather than the energy transmitted per signal, \( E \). If we let \( E_b \) be the energy transmitted per bit then these are related as follows

\[ E_b = \frac{E}{\log_2 M}. \]

Thus the bound on the symbol error probability can be expressed in terms of the energy transmitted per bit as

\[ P_{e,s} \leq \begin{cases} 1, & \frac{E_b}{N_0} \leq \ln 2 \\ \exp_2 \left\{ -\log_2 M \left( \sqrt{\frac{E_b}{N_0}} - \sqrt{\ln 2} \right) \right\}, & \ln 2 \leq \frac{E_b}{N_0} \leq 4\ln 2 \\ \exp_2 \left\{ -\log_2 M \left( \frac{E_b}{4N_0} - \ln 2 \right) \right\}, & \frac{E_b}{N_0} \geq 4\ln 2 \end{cases}. \]

where \( \exp_2(x) \) denotes \( 2^x \). Note that as \( M \to \infty, P_e \to 0 \) if \( \frac{E_b}{N_0} > \ln 2 = -1.59\text{dB} \).
So far we have examined the symbol error probability for orthogonal signals. Usually the number of such signals is a power of 2, e.g., 4, 8, 16, 32, ... If so then each transmission of a signal is carrying \( k = \log_2 M \) bits of information. In this case a communication engineer is usually interested in the bit error probability as opposed to the symbol error probability. These can be related for any equidistant, equienergy signal set (such as orthogonal or simplex signal sets) by

\[
P_{e,b} = \frac{2^k}{2^{k-1}} P_{e,s} = \frac{M}{2(M-1)} P_{e,s},
\]

Figure 77: Symbol Error Probability for Coherent Demodulation of Orthogonal Signals

Figure 78: Bit Error Probability for Coherent Demodulation of Orthogonal Signals

General Noncoherent Demodulator

Choose Largest
If signal 1 is transmitted during the interval \([(l - 1)T, lT)\) then

\[
X_{c,m}(lT) = \begin{cases} 
\sqrt{E} \cos(\theta) + \eta_{c,1}, & m = 1 \\
\eta_{c,m}, & m \neq 1
\end{cases}
\]

\[
X_{s,m}(lT) = \begin{cases} 
\sqrt{E} \sin(\theta) + \eta_{s,1}, & m = 1 \\
\eta_{s,m}, & m \neq 1
\end{cases}
\]

The decision statistic then (if signal 1 is transmitted) has the form

\[
Z_1(lT) = E + 2\sqrt{E}(\eta_{c,1} \cos(\theta) + \eta_{s,1} \sin(\theta)) + \eta_{c,1}^2 + \eta_{s,1}^2
\]

\[
Z_2(lT) = \eta_{c,2}^2 + \eta_{s,2}^2
\]

\[
Z_M(lT) = \eta_{c,M}^2 + \eta_{s,M}^2
\]

The symbol error probability for noncoherently detection of orthogonal signals is

\[
P_{e,s} = \frac{1}{M} \left[ E_b / (\log_2 MN_0) \right] \sum_{m=2}^{M} (-1)^{m} \left( \frac{M}{m} \right) e^{E_b / (m \log_2 MN_0)}
\]

As with coherent demodulation the relation between bit error probability and symbol error probability for noncoherent demodulation of orthogonal signals is

\[
P_{e,b} = \frac{2^{k-1}}{2^k-1} P_{e,s} = \frac{M}{2(M-1)} P_{e,s}
\]

Figure 79: Symbol Error Probability for Noncoherent Detection of Orthogonal Signals.

Figure 80: Bit Error Probability of $M$-ary orthogonal modulation in an additive white Gaussian noise channel with noncoherent demodulation.
A. Time-orthogonal (Pulse position modulation PPM)

\[ \psi_i(t) = \begin{cases} \sqrt{\frac{2M}{T}} \sin(2\pi f_0 t), & \frac{2iT}{M} \leq t < \frac{i+1}{M}T/M \\ 0, & \text{elsewhere} \end{cases} \]

\[ i = 0, 1, \ldots, M-1, \quad f_0 = \frac{nM}{2T} \]

\[ \int_{iT/M}^{(i+1)T/M} \sqrt{\frac{2M}{T}} \sin(2\pi f_0 t) \sqrt{\frac{2M}{T}} \sin(2\pi f_1 t) dt = 0 \]

\[ f_0 = \frac{nM}{2T}, \quad f_1 = (n+1) \frac{M}{2T}. \]

B. Time-orthogonal quadrature-phase

\[ \psi_2(t) = \begin{cases} \sqrt{\frac{2M}{T}} \sin(2\pi f_0 t), & \frac{2iT}{M} \leq t < \frac{i+1}{M}T/M \\ 0, & \text{elsewhere} \end{cases} \]

\[ \psi_{2i+1}(t) = \begin{cases} \sqrt{\frac{2M}{T}} \cos(2\pi f_0 t), & \frac{2iT}{M} \leq t < \frac{i+1}{M}T/M \\ 0, & \text{elsewhere} \end{cases} \]

\[ i = 0, 1, \ldots, \frac{M}{2} - 1, \quad \text{M even,} \quad f_0 = \frac{nM}{2T} \]

C. Frequency-orthogonal (Frequency Shift Keying FSK)

\[ \psi_i(t) = \sqrt{\frac{2E}{T}} \sin[2\pi(f_0 + \frac{i}{2T})t], \quad 0 \leq t \leq T \]

\[ i = 0, 1, \ldots, M-1, \quad f_0 = \frac{nM}{2T}. \]

D. Frequency-orthogonal quadrature-phase

\[ \psi_2(t) = \sqrt{\frac{2E}{T}} \sin[2\pi(f_0 + \frac{i}{2T})t], \quad 0 \leq t \leq T \]

\[ \psi_{2i+1}(t) = \sqrt{\frac{2E}{T}} \cos[2\pi(f_0 + \frac{i}{2T})t], \quad 0 \leq t \leq T \]

\[ f_0 = \frac{nM}{2T}. \]
E. Hadamard-Walsh Construction

The last construction of orthogonal signals is done via the Hadamard Matrix. The Hadamard matrix is an \(N\) by \(N\) matrix with components either +1 or -1 such that every pair of distinct rows are orthogonal. We show how to construct a Hadamard when the number of signals is a power of 2 (which is often the case).

Begin with a two by two matrix

\[
H_2 = \begin{bmatrix}
+1 & +1 \\
+1 & -1
\end{bmatrix}.
\]

Then use the recursion

\[
H_{2^l} = \begin{bmatrix}
H_{2^{l-1}} & H_{2^{l-1}} \\
H_{2^{l-1}} & -H_{2^{l-1}}
\end{bmatrix}.
\]

Now it is easy to check that distinct rows in these matrices are orthogonal. The \(i\)-th modulated signal is then obtained by using a single (arbitrary) waveform \(N\) times in nonoverlapping time intervals and multiplying by the \(j\)-th repetition of the waveform by the \(j\)-th component of the \(i\)-th row of the matrix.

Example \((M = 4)\):

\[
H_4 = \begin{bmatrix}
H_2 & H_2 \\
H_2 & -H_2
\end{bmatrix} = \begin{bmatrix}
+1 & +1 & +1 & +1 \\
+1 & -1 & +1 & -1 \\
+1 & +1 & -1 & +1 \\
+1 & -1 & -1 & -1
\end{bmatrix}.
\]

Example \((M = 8)\):

\[
H_8 = \begin{bmatrix}
H_4 & H_4 \\
H_4 & -H_4
\end{bmatrix} = \begin{bmatrix}
H_2 & H_2 & H_2 & H_2 \\
H_2 & -H_2 & H_2 & -H_2 \\
H_2 & H_2 & -H_2 & -H_2 \\
H_2 & -H_2 & -H_2 & H_2
\end{bmatrix}.
\]
Noncoherent Reception of Hadamard Generated Orthogonal Signals

\[
W_1 = (X_1 + X_2 + X_3 + X_4 + X_6 + X_7 + X_8)^2 + (Y_1 + Y_2 + Y_3 + Y_4 + Y_6 + Y_7 + Y_8)^2
\]

\[
W_2 = (X_1 - X_2 - X_4 - X_5 - X_6 - X_7 - X_8)^2 + (Y_1 - Y_2 - Y_3 - Y_4 - Y_6 - Y_7 - Y_8)^2
\]

\[
W_3 = (X_1 + X_2 - X_3 - X_4 + X_5 + X_6 + X_7 + X_8)^2 + (Y_1 + Y_2 - Y_3 - Y_4 + Y_5 + Y_6 + Y_7 - Y_8)^2
\]

\[
W_4 = (X_1 - X_2 - X_3 + X_4 + X_5 - X_6 + X_7 - X_8)^2 + (Y_1 - Y_2 + Y_3 - Y_4 + Y_5 - Y_6 - Y_7 + Y_8)^2
\]

Figure 82: Noncoherent Demodulator

Figure 83: Fast Processing for Hadamard Signals
If we define bandwidth of $M$ signals as minimum frequency separation between two such signal sets such that any signal from one signal set is orthogonal to every signal from a frequency adjacent signal set are orthogonal then for all of these examples of $M$ signals the bandwidth is

$$W = \frac{M}{2T} \Rightarrow M = 2WT!$$

Thus there are $2WT$ orthogonal signals in bandwidth $W$ and time duration $T$. 

---

Biorthogonal Signal Set

A biorthogonal signal set can be described as

$$s_0(t) = \sqrt{E} \phi_0(t)$$
$$s_1(t) = \sqrt{E} \phi_1(t)$$
$$\ldots$$
$$s_{M/2-1}(t) = \sqrt{E} \phi_{M/2-1}(t)$$
$$s_M(t) = -\sqrt{E} \phi_0(t)$$
$$\ldots$$
$$s_{M+1}(t) = -\sqrt{E} \phi_{M/2-1}(t)$$

That is a biorthogonal signal set is the same as orthogonal signal set except that the negative of each orthonormal signal is also allowed. Thus there are $2N$ signals in $N$ dimensions. We have doubled the number of signals without changing the minimum Euclidean distance of the

---

Symbol Error Probability

Let $H_j$ be the hypothesis that signal $s_j$ was sent for $j = 0, \ldots, M-1$. The probability of correct is (given signal $s_0$ sent)

$$P_{c,0} = P(r_0 > 0, |r_1| < 0, \ldots, |r_{M/2-1}| < 0|H_0)$$

$$= \int_{r_0=0}^{\infty} f_r(r_0)[F_n(r_0) - F_n(-r_0)]^{M/2} \cdot 1 \cdot dr_0$$

where $f_r(x)$ is the density function of $r_0$ when $H_0$ is true and $F_n(x)$ is the distribution of $r_1$ when $H_0$ is true.

$$f_r(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(x - \sqrt{E})^2\right)$$
$$F_r(x) = \Phi\left(\frac{x - \sqrt{E}}{\sigma}\right)$$
$$f_n(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(x)^2\right)$$
$$F_n(x) = \Phi\left(\frac{x}{\sigma}\right)$$

---
where $\sigma^2 = N_0/2$. The error probability is then

$$ P_{e,0} = 1 - \int_{r_0=0}^{\infty} f_z(r_0)[F_n(r_0) - F_n(-r_0)]^{M/2} \, dr_0 $$

Using an integration by parts argument we can write this as

$$ P_{e,s} = (M - 2) \int_0^\infty [F_z(r_j) - F_z(-r_j)][F_n(r_j) - F_n(-r_j)]^{M/2} \, f_z(r_j) \, dr_j $$

It should be obvious that this is also the error probability to $H_j$ for $j > M/2$. The probability of error of the second kind is the probability that $H_{M/2}$ is chosen given that $s_0$ is transmitted and is given by

$$ P_{e,2} = P(r_0 < 0, |r_1| < |r_0|, |r_2| < |r_0|, ..., |r_{M/2} < |r_0|) \, |H_0|_2 $$

$$ = \int_0^\infty f_z(r_0)[F_n(r_0) - F_n(-r_0)]^{M/2} \, dr_0 $$

$$ = (M - 2) \int_0^\infty [F_z(r_j) - F_z(-r_j)][F_n(r_j) - F_n(-r_j)]^{M/2} \, f_z(r_j) \, dr_j $$

The bit error probability is determined by realizing that of the $M - 2$ possible errors (all equally likely) of the first kind, $(M - 2)/2$ of them result in a particular bit in error while an error of the second kind causes all the bits to be in error. Thus

$$ P_{e,b} = \frac{M - 2}{2} P_{e,1} + P_{e,2} $$

$$ = \frac{(M - 2)}{2} \int_0^\infty [F_z(u) + F_z(-u)][F_n(u) - F_n(-u)]^{M/2} \, f_z(u) \, du $$

**Bit Error Probability**

The bit error probability for biorthogonal signals can be determined for the usual mapping of bits to symbols. The mapping is given as

<table>
<thead>
<tr>
<th>$s_0(t)$</th>
<th>$s_1(t)$</th>
<th>$s_{M/2}$</th>
<th>$s_{M}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>000000 ... 000</td>
<td>000000 ... 001</td>
<td>011111 ... 111</td>
<td>111111 ... 110</td>
</tr>
<tr>
<td>000000 ... 000</td>
<td>000000 ... 001</td>
<td>011111 ... 111</td>
<td>111111 ... 110</td>
</tr>
<tr>
<td>000000 ... 000</td>
<td>000000 ... 001</td>
<td>011111 ... 111</td>
<td>111111 ... 110</td>
</tr>
</tbody>
</table>

The mapping is such that signals with furthest distance have largest number of bit errors. An error of the first kind is defined to be an error to an orthogonal signal, while an error of the second kind is an error to the antipodal signal. The probability of error of the first kind is the probability that $H_j$ is chosen given that $s_0$ is transmitted ($j < M/2$) and is given by

$$ P_{e,1} = P(r_j > |r_0|, r_j > |r_1|, ..., r_j > |r_{M/2 - 1}|, r_j > 0) $$

$$ = \frac{(M - 2)}{2} \int_0^\infty [\Phi(z - \sqrt{\frac{3\sigma^2}{N_0}})]^\infty \, dz $$

Notice that the symbol error probability is $P_{s,b} = (M - 2)P_{e,1} + P_{e,2}$.
Simplex Signal Set

Same as orthogonal except subtract from each of the signals the average signal of the set, i.e.

\[ s'_i(t) = s_i(t) - \frac{1}{M} \sum_{j=0}^{M-1} s_j(t), \quad i = 0, 1, ..., M-1 \]

When the orthogonal set is constructed via a Hadamard matrix this amounts to deleting the first component in the matrix since the other components sum to zero.

For example

\[
S_8 = \begin{bmatrix}
+1 & +1 & +1 & +1 & +1 & +1 \\
-1 & +1 & -1 & +1 & -1 & -1 \\
+1 & -1 & -1 & +1 & -1 & -1 \\
-1 & -1 & +1 & -1 & -1 & +1 \\
+1 & +1 & -1 & -1 & -1 & +1 \\
-1 & +1 & -1 & +1 & +1 & -1 \\
+1 & -1 & -1 & -1 & +1 & +1 \\
-1 & -1 & +1 & -1 & +1 & -1 \\
\end{bmatrix}
\]

These are slightly more efficient than orthogonal signals.
Multiphase Shift Keying (MPSK)

\[ s_i(t) = A \cos \left( 2\pi f_c t + \frac{2\pi i}{M} + \lambda \right), \quad 0 \leq t \leq T \]
\[ = A_{ci} \cos 2\pi f_c t - A_{si} \sin 2\pi f_c t \]

where for \( i = 0, 1, \ldots, M-1 \),
\[ A_{ci} = A \cos \left( \frac{2\pi i}{M} + \lambda \right) \]
\[ A_{si} = A \sin \left( \frac{2\pi i}{M} + \lambda \right) \]

\[ P_{es} = 1 - \int_{\pi/iM}^{\pi/M} \frac{E/N_0}{2\pi} \left[ 1 + \sqrt{\frac{4\pi E}{N_0}} \cos \theta \exp \left( i \cos \theta \left( 1 - Q \left( \sqrt{\frac{2E}{N_0} \cos \theta} \right) \right) \right] d\theta \]

For this modulation scheme we should use Gray coding to map bits into signals.

\( M = 2 \Rightarrow \text{BPSK} \quad M = 4 \Rightarrow \text{QPSK} \)

This type of modulation has the properties that all signals have the same power thus the use of nonlinear amplifiers (class C amplifiers) affects each signal in the same manner. Furthermore if we are restricted to two dimensions and every signal must have the same power than this signal set minimizes the error probability of all such signal sets.

(QPSK and BPSK are special cases of this modulation).
**M-ary Pulse Amplitude Modulation (PAM)**

\[ s_i(t) = A_i s(t), \quad 0 \leq t \leq T \]

where

\[ A_i = (2i+1-M)A \quad i = 0, 1, \ldots, M - 1 \]

\[ E_i = A_i^2 \]

\[ E = \frac{1}{M} \sum_{i=0}^{M-1} E_i = \frac{A^2}{M} \sum_{i=0}^{M-1} (2i+1-M)^2 \]

\[ = \frac{M^2 - 1}{3} A^2 \]

\[ P_{e,i} = \left( \frac{2(M-1)}{M} \right) Q \left( \sqrt{\frac{6E}{(M^2-1)N_0}} \right) \]
Bandwidth of Digital Signals:

In practice a set of signals is not used once but in a periodic fashion. If a source produces symbols every T seconds from the alphabet \( A = 0, 1, \ldots, M - 1 \) with \( b_i \) representing the \( i^{th} \) letter \( \infty \leq i < \infty \) then the digital data signal has the form

\[
s(t) = \sum_{l=-\infty}^{\infty} s_{b_l}(t-nT)
\]

Note: 1) \( s_i(t) \) need not be time limited to \([0, T]\). In fact we may design \( \{s_i(t)\}_{i=0}^{M-1} \) so that \( s_i(t) \) is not time limited to \([0, T]\). If \( s_i(t) \) is not time limited to \([0, T]\) then we may have intersymbol interference in the demodulator. The reason for introducing intersymbol interference is to "shape" the spectral characteristic of the signal (e.g. if there are nonlinear amplifiers or other nonlinearities in the communication system).

2) The random variables \( b_n \) need not be a sequence of i.i.d. random variables. In fact if we are using error-correcting codes there will be some redundancy in \( b_2 \) so that it is not a sequence of i.i.d. r.v.

In many of the modulation schemes (the linear ones) considered we can equivalently write the signal as

\[
s(t) = \text{Re}[u(t)e^{j\omega_{ct}}]
\]

where \( u(t) \) is called the lowpass signal. For general CPM the modulation is nonlinear so that the below does not apply. Also

\[
u(t) = \sum_{n=0}^{\infty} I_n g(t-nT)
\]

where \( I_n \) is possibly complex and \( g(t) \) is an arbitrary pulse shape.

Note that while \( u(t) \) is a (non-stationary) random process \( u(t+\tau) \) where \( \tau \) is uniform r.v. on \([0, T]\) is stationary.

\[
\Phi_u(f) \triangleq \mathcal{F}\{E[u^*(t+\tau)u(t+\tau+\tau)]\}
= \frac{1}{T}\Phi_u(f)|G(f)|^2
\]

where

\[
\Phi_f(f) = \sum_{n=0}^{\infty} E[I_n^*I_{n+m}] e^{-j2\pi ft}
\]

\[
G(f) = \mathcal{F}\{g(t)\} = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft} dt
\]

Example: BPSK \( I_n = \pm 1 \) (i.i.d.)
\[ g(t) = A \cos(\omega_0 t + \phi(t)) \quad 0 \leq t \leq T \]
\[ E[h_{l+m}] = \delta_{m,0} = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases} \]
\[ \Phi_1(f) = 1 \]
\[ \Phi_n(f) = \frac{A^2 T}{4} \left( \text{sinc}^2(\omega - \omega_c) + \text{sinc}^2(\omega + \omega_c) \right) \]

**Definition of Bandwidth**

1. **Null-to-Null** \( \triangleq \) bandwidth (in Hz) of main lobe (= \( \frac{T}{2} \) for BPSK).
2. **99% containment bandwidth** \( \triangleq \) bandwidth such that = \( \frac{3}{2} \)% lies above upper bandlimit = \( \frac{1}{2} \)% lies below lower level.
3. **x dB bandwidth** \( W_x \) \( \triangleq \) bandwidth such that spectrum is \( x \) dB below spectrum at center of band (e.g. 3dB bandwidth).

**Comparison of Modulation Techniques**

BPSK has \( P_e, d = \frac{\sqrt{\omega}}{2N_0} \)

\[ W = \frac{1}{T}, \quad R = \frac{1}{T}, \quad \Rightarrow \frac{R}{W} = 1.0 \]

\[ E_b / N_0 = 9.6dB \]

QPSK has same \( P_e \) but has \( \frac{R}{W} = 2.0 \)

\[ R = \frac{3}{T} \quad (R = \frac{1}{T}) \]

or

\[ W = \frac{1}{T} \quad (W = \frac{1}{2T}) \]
M-ary PSK has same bandwidth as BPSK but transmits \( \log_2 M \) bits/channel use (T sec).

M-ary PSK

\[
R = \frac{\log_2 M}{T} \quad \Rightarrow \quad \frac{R}{W} = \log_2 M
\]

\[
W = \frac{1}{T}
\]

Capacity (Shannon Limits)

\[
\frac{R}{W} < \log_2 \left( 1 + \frac{R E_b}{W N_0} \right)
\]

or

\[
\frac{E_b}{N_0} > \frac{(2^{R/W} - 1)}{R/W}
\]

We can come close to capacity (at fixed \( \frac{R}{W} \)) by use of coding (At \( \frac{R}{W} = 1 \) there is a possible 9.6 dB "coding gain")