QUANTUM THEORY OF LIGHT
EECS 638/PHYS 542/AP609
FINAL EXAMINATION

Instructor: Professor S.C. Rand
Date: April 25, 2001
Duration: 2.5 hours

PLEASE read over the entire examination before you start. DO ALL QUESTIONS and show all your work in submitted material to be eligible for partial credit.

Useful formulae:

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cos(2x)=cos^2(x)-sin^2(x)
sin(2x)=2sin(x)cos(x)

The Pauli spin matrices:

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \sigma^{\pm} = \frac{1}{2}(\sigma_x \pm i\sigma_y)$$

Honor Code: I have neither given nor received aid on this exam.

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Name (Print)

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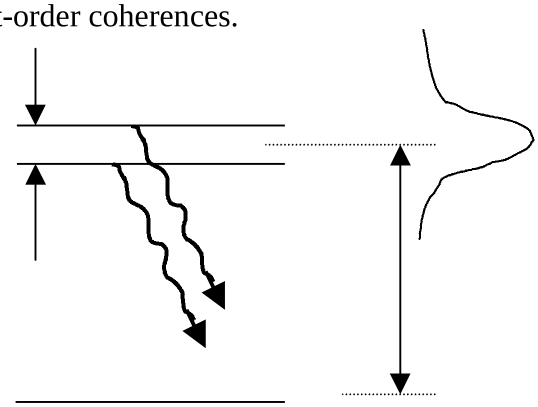
Signature

1. **(15 marks total)**
   A system with two closely-spaced excited levels $|2\rangle$ and $|3\rangle$ lying above the ground state $|1\rangle$ has allowed transitions $|1\rangle \leftrightarrow |2\rangle$ and $|1\rangle \leftrightarrow |3\rangle$, with corresponding dipole moments $\mu_{12} \neq 0$ and $\mu_{13} \neq 0$. Assume $\mu_{23}=0$. The energy separation of $|2\rangle$ and $|3\rangle$ is $\Delta\omega = \omega_3 - \omega_2$. An ultrashort pulse of light with a frequency bandwidth in excess of $\Delta\omega$ excites the system at a carrier frequency ω_L at time t=0 ($\omega_L \cong \omega_2 - \omega_1 \cong \omega_3 - \omega_1$), generating several first-order coherences.



(a) With reference to the nine density matrix elements of this 3-level system, identify the non-zero, first-order coherences. **(3 marks)**

(b) Give a formal expression for the microscopic polarization field p as a trace over appropriate operators, and write out the non-zero terms explicitly. **(3 marks)**

(c) Given the microscopic polarization of the medium, from what equations must the radiative signal field E_s be calculated in general to account for macroscopic properties of the medium or the field, and what is the value of E_s here (assuming for simplicity that there are no inhomogeneities such as position-dependent saturation to take into account)? **(2 marks)**

Problem 1 (Cont'd):

(d) Since individual coherences here are excited by light with considerable bandwidth, the frequency of their oscillation is not obvious. What is the frequency of each coherence? Justify by explicit reference to discussion or figures in course notes**. (2 marks)**

(e) Show that the intensity of spontaneous emission of this system contains a term (for $t>0$) oscillating at the level splitting frequency $\Delta\omega$. (This provides a method of measuring small excited-state splittings). **(5 marks)**

2. **(15 marks total)**

The interaction Hamiltonian of a quantized field mode interacting with a two-level atom, in the rotating wave approximation, is

$$H = H_0 + V = \frac{\hbar\omega_0}{2}\sigma_z + \hbar\omega_L a^+ a^- + \hbar g(\sigma^+ a^- + a^+ \sigma^-) \ ,$$

where $a^{\pm}$ and $\sigma^{\pm}$ are raising and lowering operators of the field and the atom respectively. $\sigma_i \, (i = 1,2,3)$ are the Pauli spin matrices (see useful formulae), $\hbar\omega_0$ is the atomic energy level separation and $\hbar\omega_L$ is the incident laser photon energy.

(a) Verify by direct substitution that, for an intense field applied on resonance, the states

$$\left|\pm, n\right\rangle = \frac{1}{\sqrt{2}}\left(\left|a, n\right\rangle \pm \left|b, n+1\right\rangle\right)$$

are eigenstates of the combined atom-field system. **(4 marks)**

(b) Identify the corresponding eigenvalues. **(2 marks)**

Problem 2 (Cont'd):

(c) Show that electric-dipole transitions involving the $|+,n\rangle$ state are not allowed *at the Rabi frequency* $\Omega_n = 2g\sqrt{n+1}$. **(2 marks)**

(d) Suppose that the laser is modulated* to produce two frequency components at ω_L and (ω_L - Ω_n). Draw a diagram in the dressed state picture of a 2-photon transition $|-,n\rangle \rightarrow |+,n\rangle$ utilizing one photon at each frequency. **(1 mark)**

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* Assume the modulation frequency falls within the single mode bandwidth (Ω<<cΔk), so the problem still involves only the original mode.

Problem 2 (Cont'd):

(e) Determine whether the 2-photon transition of part (d) is allowed.
(4 marks)

(f) For the sake of argument, suppose the frequency component of the modulated laser field at (ω_L - Ω_n) is intense, like the component at ω_L, and that the $|-,n\rangle \rightarrow |+,n\rangle$ transition can be strongly driven on resonance by the 2-photon transition. Could the quasi-stationary states $|-,n\rangle$ and $|+,n\rangle$ of the dressed atom themselves each split under these circumstances? Why or why not? **(2 marks)**

3. General solutions for the probability amplitudes $c_{a,n}(t)$ and $c_{b,n+1}(t)$ of a 2-level atom interacting with a single mode light field (where the atom may be in the excited state with n photons present or in the ground state with n+1 photons respectively) are:

$$c_{a,n}(t) = \left\{ c_{a,n}(0)\left[\cos\left(\frac{\Omega_n t}{2}\right) - \frac{i\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right] - \frac{2ig\sqrt{n+1}}{\Omega_n} c_{b,n+1}(0)\sin\left(\frac{\Omega_n t}{2}\right)\right\} e^{i\Delta t/2},$$

and

$$c_{b,n+1}(t) = \left\{ c_{b,n+1}(0)\left[\cos\left(\frac{\Omega_n t}{2}\right) + \frac{i\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right] - \frac{2ig\sqrt{n+1}}{\Omega_n} c_{a,n}(0)\sin\left(\frac{\Omega_n t}{2}\right)\right\} e^{-i\Delta t/2},$$

$\Omega_n^2 = \Delta^2 + 4g^2(n+1)$ is the square of the Rabi frequency at detuning Δ for a field containing n photons in the one mode, and g is the transition coupling constant. Assume the atom is initially in the excited state, so that $c_{a,n}(0) = c_n(0)$ and $c_{b,n+1}(0) = 0$, where $c_n(0)$ is the probability amplitude that the field contains n photons.

(a) Write out simplified probability amplitudes for $c_{a,n}(t)$ and $c_{b,n+1}(t)$ using the initial conditions and calculate $|c_{a,n}(t)|^2$ and $|c_{b,n}(t)|^2$. **(3 marks)**

(b) Using the results of part (a), calculate the population inversion W(t), given by the excited state population density minus the ground state population density, summed over photon number n. **(1 mark)**

Problem 3 (Cont'd):

(c) Define the probability of n photons being present at t=0 as $\rho_{nn}(0) \equiv \left|c_n(0)\right|^2$,
and show that W(t) can be rearranged to yield the simple but exact expression

$$W(t) = \sum_{n=0}^{\infty} \rho_{nn}(0)\left[\frac{\Delta^2}{\Omega_n^2} + \frac{4g^2(n+1)}{\Omega_n^2}\cos(\Omega_n t)\right] \qquad (1)$$

with no approximations. **(4 marks)**

(d) Explain why the overall inversion varies sinusoidally at a frequency different from that of the light. **(2 marks)**

(e) What is W for the "vacuum" field ? **(1 mark)**

Problem 3 (Cont'd):

(f) If W for the "vacuum" field is time-invariant, explain why physically. If it is time-dependent, explain why. **(2 marks)**

(g) In the Weisskopf-Wigner theory of spontaneous emission, the inversion undergoes exponential decay due to transitions mediated by the vacuum field. What is the essential difference between Wigner-Weisskopf theory and the development of Eq.(1) above? **(1 mark)**

(h) Can you suggest an experiment that might permit the observation of the vacuum field behavior predicted by Eq.(1)? **(1 mark)**