Quantum Theory of Light
EECS 638/PHYS542/AP609
Midterm Exam

Instructor: Professor S.C. Rand
Date: March 6, 2001
Duration: 1.5 hours

Useful Formulae:

Vector model evolution matrices for in the case of

(i) Nutation:

\[
\begin{bmatrix}
\frac{\chi^2 + \Delta^2 \cos \theta}{\Omega^2} & -\Delta \sin \theta \Omega^{-1} & -\frac{\Delta \chi}{\Omega} (1 - \cos \theta) \\
\frac{\Delta}{\Omega} \sin \theta & \cos \theta & \frac{\chi}{\Omega R} \sin \theta \\
-\frac{\Delta \chi}{\Omega} (1 - \cos \theta) & -\frac{\chi}{\Omega R} \sin \theta & \frac{\Delta^2 + \chi^2 \cos \theta}{\Omega^2}
\end{bmatrix}
\]

\[\theta = \chi \tau; \; \chi = \frac{\mu E}{\hbar}; \; \Omega_R = \sqrt{\Delta^2 + \chi^2}\]

(ii) Free induction decay:

\[
\begin{bmatrix}
e^{\frac{(t-t_0)}{T_1}} \cos \Delta (t-t_0) & e^{\frac{(t-t_0)}{T_2}} \sin \Delta (t-t_0) & 0 \\
e^{\frac{(t-t_0)}{T_2}} \sin \Delta (t-t_0) & e^{\frac{(t-t_0)}{T_1}} \cos \Delta (t-t_0) & 0 \\
0 & 0 & e^{\frac{(t-t_0)}{T_1}}
\end{bmatrix}
\]

\[\Delta = \omega_0 - \omega\]

\[R_1 = \tilde{\rho}_{21} + \tilde{\rho}_{12}\]
\[R_2 = i(\tilde{\rho}_{21} - \tilde{\rho}_{12})\]
\[R_3 = \rho_{22} - \rho_{11}\]

Name ____________________________________________

Signature ___________________________________________
1. (25 marks total)

(a) Write down the equation of motion for the density matrix of a closed 2-level system, including a formal term to represent decay. (1 mark)

(b) Write out the equations for the time dependence of $\rho_{11}$, $\rho_{22}$, $\rho_{12}$ and $\rho_{21}$ assuming that a light field $E(t) = \frac{1}{2} E_0 e^{i\omega t} + c.c.$ couples ground level $|1> \to |2>$ via a transition dipole moment $\mu$, and that de-excitation of level $|2>$ occurs by spontaneous decay. (8 marks)

(c) Assume in the case of resonant optical excitation that steady-state, oscillatory solutions for the level populations exist, of the form

$$\rho_{11} = \tilde{\rho}_{11} (1 - \cos \xi t)$$
$$\rho_{22} = \tilde{\rho}_{22} (1 + \cos \xi t)$$

Use closure to find consistent values for $\tilde{\rho}_{11}$ and $\tilde{\rho}_{22}$. (Note: Do not try to derive the population expressions above - take them to be given). (4 marks)

(d) Find the slowly varying amplitude $\tilde{\rho}_{12}$ of the polarization $\rho_{12} = \tilde{\rho}_{12} e^{i\omega t}$. (7 marks)

(e) What would you expect the value of $\xi$ to be, and why does $\tilde{\rho}_{12}$ periodically go to zero (what is the physical reason)? (4 marks)

(f) Does the absorption of the system go to zero periodically (show and explain)? (1 marks)
2. **(25 marks total)**

(a) Give a general analytic expression for the macroscopic, radiative polarization \( P \) inside a 2-level system in terms of the trace of relevant operators and then evaluate it in terms of specific matrix elements assuming there is an allowed transition with a transition dipole moment \( \mu \) between the two levels. (3 marks)

(b) Find the Bloch vector of a 2-level system, initially in the ground state, that is irradiated by a resonant \( \pi \)-pulse of short duration \( \tau_p = \tau \) \((\tau << T_1, \tau >> T_2)\). Take the time of observation to be twice the pulse duration \((t=2\tau)\). (7 marks)

(c) Calculate and explain the value of polarization \( P \) observed at time \( t=2\tau \)? (3 marks)

(d) Draw the Bloch vector in a suitable coordinate system at times \( t=1.5\tau \) and \( t=2\tau \). (2 marks)

(e) What is the excited state population at \( t=2\tau \)? (2 marks)

(f) What is the sign of the absorption at \( t=2\tau \)? Would you expect a probe pulse to experience loss (positive absorption) or gain (negative absorption)? (2 mark)

(g) If a second pulse, a \( \frac{\pi}{2} \) pulse, is applied at time \( t=2\tau \) \((\tau << T_1, \tau >> T_2)\), what is the polarization at the end of the second pulse? (3 marks)

(h) Assuming the system to be inhomogeneously broadened, would you expect an echo at time \( t=4\tau \)? Why or why not? (3 marks)