We wish to use high-resolution analysis to find an approximate formula for the OPTA function $\delta(k,M)$ (or $\delta(k,R)$) for arbitrary $k$ and large $M$ (or $R$). (Recall that $\delta(k,M)$ is defined to be the distortion of an optimal VQ with dimension $k$ and size $M$.) In this section, we consider fixed-length quantization. Later we consider variable-length quantization.

Zador’s Theorem 1: When quantizing a random vector $X$, 

$$M^2 \delta(k,M) \rightarrow \frac{1}{M^{2/k}} m^*_k \left( \int f_X(x)^{k/(k+2)} \, dx \right)^{(k+2)/k}$$

as $M \rightarrow \infty$. Therefore, when $M$ is large, we have the following approximation:

$$\delta(k,M) \simeq \frac{1}{M^{2/k}} m^*_k \left( \int f_X(x)^{k/(k+2)} \, dx \right)^{(k+2)/k}$$

where $f_X$ is the pdf of the random vector $X$ being quantized, and $m^*_k$ is a constant, to be described later, that depends on the dimension $k$ but not the pdf $f_X$.

We first discuss the derivation of the theorem and later discuss its interpretation and ramifications.

As Zador’s proof of his theorem is rather complex, we shall instead pursue an informal derivation based on Bennett’s integral.

Our goal is to find a formula for the least possible distortion of $k$-dimensional quantizers with large size $M$. Since $M$ is large, we can assume that optimal quantizers satisfy the high-resolution conditions, so that the distortion of such a quantizer can be approximated by Bennett’s integral:

$$D \simeq \frac{1}{M^{2/k}} \int \frac{m(x)}{\lambda^{2/k}(x)} f_X(x) \, dx$$

where $m$ is the inertial profile and $\lambda$ is the point density of any optimal quantizer. Our approach is to find $m$ and $\lambda$ for optimal quantizers, and then substitute them into Bennett’s integral.

A first thought is that $m$ and $\lambda$ should be functions that minimize Bennett’s integral, i.e. that minimize

$$\int \frac{m(x)}{\lambda^{2/k}(x)} f_X(x) \, dx$$

However, we cannot simply minimize the above over arbitrary functions $m$ and $\lambda$. Instead we must minimize the above over “valid” choices of $m$ and $\lambda$, where by valid choices, we mean those that could actually be the inertial profile and point density of actual quantizers. For example, for any actual quantizer

$$m(x) \geq nmi(k\text{-dimensional sphere}) \text{ for all } x$$
So clearly any \( m(x) \) that does not satisfy the above is invalid.

The question becomes what choices of \( m \) and \( \lambda \) are valid? This is a difficult question that has not been completely answered. Consequently, we take a less direct approach.

We first argue that the inertial profile of any optimal quantizer is a constant, i.e. it does not vary with \( x \). We will let \( m^*_k \) denote this constant.

Then we argue that for a constant inertial profile, the point density can be any nonnegative function that integrates to one. We then show how to choose \( \lambda \) to minimize Bennett’s integral, assuming a constant inertial profile. Substituting these choice for \( m \) and \( \lambda \) into Bennett’s integral will give a formula for the distortion of an optimal quantizer, which turns out to be Zador’s formula.

---

**Best Inertial Profile**

**Definition:** For \( k \)-dimensional quantization, a function \( m(x) : \mathbb{R}^k \rightarrow [0, \infty) \) is said to be a valid inertial profile if for all sufficiently large \( M \) there are quantizers with dimension \( k \), size \( M \), and inertial profile \( m \).

**Definition:** The \( k \)-dimensional shape factor\(^2\), denoted \( m^*_k \), is the minimum value of any valid inertial profile for any point density. That is,

\[
m^*_k = \min_{\text{valid } k \text{-dimensional inertial profiles } m} \min_x m(x)
\]

Later we’ll discuss what’s known about the values of find \( m^*_k \).

**Fact:** Let \( f_X \) be the pdf of a \( k \)-dimensional random vector \( X \). Then for all sufficiently large integers \( M \), any optimal \( k \)-dimensional quantizer for \( f \) with size \( M \) has inertial profile

\[
m(x) = m^*_k \text{ for all } x
\]

Note: Among other things, this indicates that best inertial profile is a constant, i.e. it does not vary with \( x \). Also, it does not depend on the probability distribution of \( X \).

Note: As we discuss later, finding the value of \( m^*_k \) is not easy.

---

\(^2\)Let me know if you think of a better name.
Sketch of Proof³:

We'll argue that if a quantizer's inertial profile \( m(x) \) does not approximately equal \( m_k^* \) for all \( x \), then the quantizer can be improved, i.e., it is does not have the smallest possible MSE for the given \( k, M \) and \( f_X \). The contrapositive of this statement is: If a quantizer has smallest MSE, then its inertial profile equals \( m_k^* \), which is the desired fact.

Accordingly, consider a \( k \)-dimensional quantizer \( Q \) with large size \( M \), whose inertial profile \( m(x) \) is greater than \( m_k^* \) at some point \( x_b \). Let \( \lambda \) denote its point density. We assume that \( m \) and \( \lambda \) are continuous or piecewise continuous functions⁴. Since \( m \) is continuous or piecewise continuous, there exists a small region \( B \) including \( x_b \) such that \( m(x) \approx m(x_b) > m_k^* \), for all \( x \in B \).

The idea now is to improve the quantizer by replacing the configuration of points and cells in \( B \) with those from a quantizer whose inertial profile equals \( m_k^* \) in some region. By the definition of \( m_k^* \), there must exist a quantizer \( Q' \) with large size whose inertial profile \( m'(x) \) equals \( m_k^* \) at some point \( x_g \). ("b" is for "bad" and "g" is for "good".)

We will assume that \( M \) is so large that the number \( M_b \) of quantization points/cells in \( B \) is itself large, and can be approximated in terms of the quantization density as \( M_b \approx \lambda(x_b) |B|M \).

³This fact has never been rigorously stated and proved.
⁴Inertial profiles and point densities are supposed to be fairly smooth.

We now wish to designate \( M_b \) quantization points/cells from \( Q' \) for "copying" into \( Q \). Let \( G \) be a region including \( x_g \) that is congruent to \( B \) and that contains \( M_b \) points/cells. We will assume that \( G \) is so small that \( m'(x) \approx m_k^* \) for \( x \in G \). We can assume this because we could have chosen \( B \) very small and \( M \) very, very large.

We now "copy" the points and cells in \( G \) and translate and scale them as group so they just fit into \( B \), replacing the original points and cells in \( B \). (We don't worry about cells on the boundary of \( G \) because they constitute a negligible fraction.) The result is that within \( B \), we now have the same number of points as originally (and consequently the same point density), but we have cells with normalized moment of inertia approximately \( m_k^* \), which is smaller than before.

To see that this has the desired effect, consider the distortion in the region \( B \), which is the only affected region. Originally, the distortion in \( B \) was

\[
\int_B \frac{m(x)}{M^2 k \lambda^{2/k}(x)} f_X(x) \, dx \approx \int_B \frac{m(x_b)}{M^2 k \lambda^{2/k}(x_b)} f_X(x) \, dx \approx \frac{m(x_b)}{M^2 k \lambda^{2/k}(x_b)} f_X(x_b) |B|
\]

After the change, the distortion in \( B \) is

\[
\int_B \frac{m_k^*}{M^2 k \lambda^{2/k}(x)} f_X(x) \, dx \approx \int_B \frac{m_k^*}{M^2 k \lambda^{2/k}(x_b)} f_X(x) \, dx \approx \frac{m_k^*}{M^2 k \lambda^{2/k}(x_b)} f_X(x_b) |B|
\]

Since \( m(x_b) > m_k^* \), the new quantizer has less distortion. Therefore, the original quantizer could not have been optimal, which is the contrapositive of the desired fact.
**Best Point Density**

Having found that the best inertial profile is the constant \( m^*_k \), we can now say that when \( M \) is large

\[
\delta(k,M) \equiv \min_{\lambda} \frac{m^*_k}{M^{2/k}} \int \frac{1}{\lambda^{2/k}(x)} f_X(x) \, dx
\]

where the minimum is take over all point densities that are valid with a constant inertial profile. We assert, without formal proof, the intuitive fact that any function \( \lambda \) that is nonnegative and integrates to one is a valid point density to use with a constant inertial profile, such as \( m(x) = m^*_k \).

It then follows that the best point density is that which minimizes the integral.

\[
\int \frac{f_X(x)}{\lambda^{2/k}(x)} \, dx
\]

(\(^*\))

Calculus of variations is one way to find \( \lambda \) that minimizes this integral subject to the negativity and integrate-to-one constraints. But we will use Holder's inequality.

---

**Holder's inequality:**

Given function's \( f \) and \( g \), then for any \( q, r > 0 \) such that \( \frac{1}{q} + \frac{1}{r} = 1 \),

\[
\int |f(x)| \, g(x) \, dx \leq \left( \int |f(x)|^q \, dx \right)^{1/q} \left( \int |g(x)|^r \, dx \right)^{1/r}
\]

with equality iff for some \( c \), \( |f(x)|^q = c \, |g(x)|^r \), all \( x \)

Our Strategy: Choose \( f, g, q \) and \( r \) so that

\[
|f(x)|^q = \frac{f_X(x)}{\lambda(x)^{2/k}} \quad \text{and} \quad \int |g(x)|^r \, dx = 1.
\]

Then Holder's inequality implies

\[
\int \frac{f_X(x)}{\lambda(x)^{2/k}} \, dx = \int |f(x)|^q \, dx \geq \left( \int |f(x)|^q \, g(x) \, dx \right) \left( \int |g(x)|^r \, dx \right)^{q/r} = \left( \int |f(x) \, g(x)| \, dx \right)^q
\]

with equality if and only if there is constant \( c \) such that \( |f(x)|^q = c \, |g(x)|^r \).

If it turns out that the integral on the far right does not depend on \( \lambda \), then we have a lower bound to the integral we are minimizing. And we can minimize the integral by choosing \( \lambda \) to satisfy the condition that gives equality in the lower bound.
Our choices:

\[ q = \frac{k+2}{k}, \quad r = \frac{k+2}{2} \quad \Rightarrow \quad \frac{1}{q} + \frac{1}{r} = \frac{k}{k+2} + \frac{2}{k+2} = 1 \]

\[ f(x) = \left( \frac{f_X(x)}{\lambda^{2/(k+2)}(x)} \right)^{k/(k+2)} \] \quad and \quad \( g(x) = \lambda^{2/(k+2)}(x) \)

Then as desired,

\[ |f(x)|^q = \frac{f_X(x)}{\lambda^{2/(k+2)}(x)} \quad \text{and} \quad \left( \int |g(x)|^r \, dx \right)^{1/r} = \left( \int \lambda(x) \, dx \right)^{1/r} = 1 \]

Therefore,

\[ \int \frac{f_X(x)}{\lambda^{2/(k+2)}(x)} \, dx \geq \left( \int |f(x)| g(x) \, dx \right)^q = \left( \int \frac{\lambda^{k/(k+2)}(x)}{\lambda} \, dx \right)^{(k+2)/k} \]

where, fortunately, the right-hand side does not depend on \( \lambda \), and where equality holds iff there is a constant \( c \) such that

\[ \frac{f_X(x)}{\lambda^{2/(k+2)}(x)} = c \lambda(x), \quad \text{i.e.} \quad \lambda(x) = c' f_X^{k/(k+2)}(x) \]

where \( c' \) is chosen to make \( \lambda(x) \) integrate to one. We conclude that the integral (*) is minimized by the point density

\[ \lambda_k^*(x) = \frac{f_X^{k/(k+2)}(x)}{\int f_X^{k/(k+2)}(x') \, dx'} \]

and the resulting minimum value is

\[ \left( \int f_X^{k/(k+2)}(x) \, dx \right)^{(k+2)/k} \]

Having shown that the optimal quantizers with dimension \( k \) have inertial profile \( m(x) \equiv m_k^{*} \) and quantization density \( \lambda(x) \equiv \lambda_k^{*} \), we substitute these into Bennett’s integral to obtain the distortion of an optimal quantizer with dimension \( k \) and large size \( M \) for a random vector \( X \) with pdf \( f_X \). That is, we find the following approximate formula for the opta function:

\[ \delta(k,M) \equiv \frac{1}{M^{2/k}} \int \frac{m_k^*}{\lambda_k^{2/k}(x)} \, f_X(x) \, dx \]

\[ = \frac{1}{M^{2/k}} \ m_k^* \int \frac{1}{(f_X^{k/(k+2)}(x)/c)^{2/k}} \, f_X(x) \, dx \quad \text{, where} \quad c = \int f_X^{k/(k+2)}(x) \, dx \]

\[ = \frac{1}{M^{2/k}} \ m_k^* \ c^{2/k} \int f_X^{k/(k+2)}(x) \, dx = \frac{1}{M^{2/k}} \ m_k^* \ c^{1+2/k} \]

\[ = \frac{1}{M^{2/k}} \ m_k^* \left( \int f_X^{k/(k+2)}(x) \, dx \right)^{(k+2)/k} \]
Zador's Theorem

When $M$ is large, the least distortion of fixed-rate, $k$-dim'l VQ with $M$ points is

$$\delta(k,M) \equiv Z(k,M)$$

where

$$Z(k,M) = \frac{1}{\sqrt{2\pi}} \left( \frac{1}{k} \sum_{i=1}^{k} \text{variance}(X_i) \right)^{(k+2)/k} \beta_k m_k^{2/k}$$

is called Zador's function

$$\sigma^2 = \text{source variance} = \frac{1}{k} \sum_{i=1}^{k} \text{variance}(X_i)$$

$$\beta_k = 1 \left( \int f_X(x)^{k/(k+2)} \, dx \right)^{(k+2)/k} = \text{"Zador's factor"}$$

$m_k^* = \text{smallest value attained by any valid inertial profile}$


Equivalent Statements of Zador's Theorem

In terms of rate:

When $R$ is large, the best fixed-rate, $k$-dimensional quantizers with rate $R$ have MSE

$$\delta(k,R) \cong \sigma^2 \beta_k m_k^* 2^{-2R} = Z(k,R)$$

In terms of SNR:

When $R$ is large, the best fixed-rate, $k$-dimensional quantizers with rate $R$ have SNR

$$S(k,R) = 10 \log_{10} \frac{\sigma^2}{Z(k,R)} = 6.02 R - 10 \log_{10} m_k^*$$

Note: SNR increases at 6 dB per bit for optimal quantizers.

Rate in terms of distortion:

When $D$ is small, the best fixed-rate, $k$-dimensional quantizers with MSE $D$ has rate

$$\gamma(k,D) \equiv \frac{1}{2} \log \left( m_k^* \sigma^2 \beta_k \right) - \frac{1}{2} \log_2(D)$$
Zador did not derive his theorem from Bennett's integral. Rather he found a direct proof of the fact that for any dimension $k$, there is a constant $\alpha_k$ such that for any probability density $f_X(x)$

$$\lim_{M \to \infty} M^{2/k} \delta(k, M) = \alpha_k \left( \int f_X(x)^{k/(k+2)} \, dx \right)^{(k+2)/k}$$

equivalently, his proof shows

$$\frac{\delta(k, M)}{Z(k, M)} \to 1 \text{ as } M \to \infty.$$  

Zador did not equate $\alpha_k$ with $m_k^*$ as we have defined it. But he did give some bounds to it.

How large must $M$ or $R$ be in order for the formulas to be accurate?

Example: Gauss-Markov Source, corr. coeff. $\rho = .9$

VQ's designed by LBG algorithm.

Straight lines are Zador's function $Z(k, R)$.

$m_4^*$ is estimated.
Data for the previous plot
Gauss-Markov Source, $\rho = .9$, SNR's in dB

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The predicted value at $R = 0$ is $-10 \log_{10} m_4^* \beta_k$

Example 2: IID Gaussian Source

VQ's designed by LBG algorithm.

Straight lines are from Zador's function $Z(k,R)$.

$m_4^*$ is estimated.
### Data for the previous plot

IID Gaussian Source, SNR’s in dB

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The predicted value at $R = 0$ is $-10 \log_{10} m_k^* \beta_k$

### Rules of Thumb:

- The Zador formula is usually accurate for $R \geq 3$ ($M \geq 2^{3k}$).
- For a given $R$, accuracy increases with dimension $k$. 
The Values of $m_k^*$

How to find the value of $m_k^*$? Suppose by some means, we can find the OPTA function $\delta(k,M)$ for some particular pdf $f_X(x)$. Then by Zador's theorem

$$\delta(k,M) \equiv \sigma^2 \beta_k m_k^* \frac{1}{M^{2/k}}$$

which can be solved for $m_k^*$:

$$m_k^* \equiv M^{2/k} \delta(k,M) \frac{1}{\sigma^2 \beta_k}$$

More precisely, suppose we can find $\lim_{M \to \infty} M^{2/k} \delta(k,M)$. Then,

$$m_k^* = \lim_{M \to \infty} M^{2/k} \delta(k,M) \frac{1}{\sigma^2 \beta_k}$$

If we wish to use this approach, what pdf should we consider? By far the easiest type of pdf to work with is a uniform pdf. That is, we take $f_X(x)$ to be a constant on some region, e.g. a cube, and zero elsewhere.

Here's what's been found with this approach:

- **k = 1**: For scalar quantizers,
  $$m_1^* = m(\text{interval}) = \frac{1}{12} = .0833$$

- **k = 2**: For two-dimensional quantizers
  $$m_2^* = m(\text{hexagon}) = 5\sqrt{3}/108 = .0802$$

This is based on the work L. Fejes Toth, which was rederived independently by D. Newman in a simpler fashion.

- **k ≥ 3**: For $k \geq 3$, this approach has not yielded an answer. Thus for $k \geq 3$, the value of $m_k^*$ is not known. However, there are several bounds, both upper and lower. Since the upper and lower bounds are fairly close, we get a pretty good approximation to $m_k^*$.

---


• Lower bound to $m^*_k$

$$m^*_k \geq m(\text{k-dimensional sphere})$$

By definition, $m^*_k$ is the least nmi of any valid inertial profile. Since no k-dimensional partition can have any cell with nmi less than that of a k-dimensional sphere, any valid inertial profile is lower bounded by the nmi of a k-dimensional sphere.

• Another lower bound is conjectured in the book *Sphere Packings, Lattices and Groups*, by J.H. Conway and N.J.A. Sloane, p. 59-62. It is tighter than the sphere lower bound.

• Upper bounds to $m^*_k$

We find upper bounds to $m^*_k$ by finding upper bounds to the OPTA function $\delta(k,M)$ of a uniform pdf.

More precisely, we find upper bounds to $\lim_{M \to \infty} M^{2/k} \delta_c(k,M)$.

How do we find upper bounds to the OPTA function?

We do this by finding the distortion of the best quantizers that we can think of for the uniform pdf. The OPTA function will be less than the distortion of any actual quantizer.
• Tesselation upper bound.

A subset $G$ of $\mathbb{R}^{2/k}$ is said to tesselate if there is a partition $S$ of $\mathbb{R}^{2/k}$, all of whose cells are translations and or rotations of $G$. (Scaling is not allowed.) The partition $S$ is said to be a tesselation. Some examples of tesselations are shown below.

Consider a $k$-dimensional pdf that is constant on a set $H$, i.e.

$$f_X(x) = \begin{cases} 1/|H|, & x \in H \\ 0, & \text{else} \end{cases}$$

and consider the quantizer whose partition $S$ is a tesselation based on the set $G$. This quantizer has quantization density

$$\lambda(x) = \begin{cases} 1/|H|, & x \in H \\ 0, & \text{else} \end{cases}$$

and inertial profile $m(x) = m(G)$. (We assume the quantization points are in the same relative position in each cell, and we suppress the notation for such points.)

From Bennett’s integral, we have

$$D \triangleq \frac{1}{M^{2/k}} \mathbb{E}_x \mathbb{E}_y \left[ \frac{m(x)}{\lambda^{2/k}(x)} \right] f_X(x) \, dx = \frac{1}{M^{2/k}} \mathbb{E}_x \mathbb{E}_y \left[ \frac{m(G)}{\lambda^{2/k}(x)} \right] \frac{1}{|H|} \, dx$$

$$= \frac{1}{M^{2/k}} m(G) |H|^{2/k}$$

$$= m(G) |G|^{2/k}, \quad \text{since} \quad |H| \equiv M |G|$$

Since the distortion of this quantizer is at least large as the OPTA function, for any tesselating cell $G$, we have

$$\sigma^2_{\beta_k} m_\star^k \frac{1}{M^{2/k}} \delta(k,M) \leq D \equiv m(G) |G|^{2/k}.$$ 

For this uniform source

$$\sigma^2_{\beta_k} = \left( \int f_X(x)^{k/(k+2)} \, dx \right)^{(k+2)/k} = \left( \int \left( \frac{1}{|H|} \right)^{k/(k+2)} \, dx \right)^{(k+2)/k} = |H|^{2/k}$$

Substituting $\sigma^2_{\beta_k}$ into the previous, rearranging and taking limit, we find

$$m_\star^k = \lim_{M \to \infty} M^{2/k} \delta(k,M) \frac{1}{\sigma^2_{\beta_k}} \leq \lim_{M \to \infty} M^{2/k} m(G) |G|^{2/k} \frac{1}{|H|^{2/k}}$$

$$= m(G) \quad \text{since} \quad |H| \equiv M |G|$$
We now conclude that \( m_k^* \) is upper bound by the nmi of the tesselating set \( G \) with least nmi, i.e.

\[
m_k^* \leq m_{T,k}^* \triangleq \min_{\text{tesselating } G} m(G)
\]

Gersho has conjectured that this bound is tight for \( k \geq 3 \). But it is not known if this true.

In fact, he conjectured that the cells of an optimal quantizer with many points are, locally, tesselations. This certainly seems to be the case for \( k = 1,2 \). For example, an LBG designed optimal \( k=2, M=256 \) quantizer for an IID pair of Gaussian variables is shown to the right.

Note that since the optimal quantizer for a nonuniform pdf will ordinarily have cells of different sizes, its partition will not be a tesselation. However, in small regions the tesselation will be apparent, i.e. locally it is approximately a tesselation.

---

- **Lattice tesselation upper bound.**

A partition \( S \) is said to be a *lattice tesselation* if it is a tesselation such that all cells are translations of each other. (Rotations are not allowed.) All lattices are tesselations, but not vice versa. Examples c. and d. given previously are lattices. The others are not.

Many examples of lattice tesselations are known. For example, see the book by Conway and Sloane. Indeed most known tesselations are lattices. By the same argument as for tesselations,

\[
m_k^* \leq m_{L,k}^* \triangleq \min_{\text{lattices } S \text{ based on } G} m(G)
\]

Of course \( m_{T,k}^* \leq m_{L,k}^* \). In fact, these might be equal, but for \( k \geq 3 \) no one knows. (For \( k=1 \) or 2, they are equal.)
• Periodic tesselations upper bound

A partition $S$ is said to be a **periodic tesselation** if there is a finite collection of, say, $M_0$ cells $G_1, \ldots, G_{M_0}$ such that the remaining cells of the partition are obtained by translating or rotating this group of cells (as a group). (Scaling is not allowed.) All of the examples given previously (a.-d.) are periodic tesselations. The following are examples of periodic tesselations that are not lattices nor ordinary tesselations.

![Periodic tesselations examples](image)

We may construct a quantizer for a uniform pdf in essentially the same way as for a tesselation. This leads to the upper bound

$$m_k^* \leq m_{PT,k}^* \triangleq \min_{\text{periodic tesselations}} \tilde{m}(G_1, \ldots, G_{M_0})$$

---

8This is a 3-dimensional periodic lattice with 2 types of cells, called the Weaire-Phelan partition. It is formed by tessellating a fundamental group consisting of two pentagonal dodecahedra (12 faces, each is 5-sided) and six 14-hedra (2 hexagonal faces, 4 pentagonal faces of one kind and 6 pentagonal faces of another kind).

where $\tilde{m}$ is the average normalized moment of inertia of a group of cells, as defined at the end of the lecture notes on Bennett's integral.

Of course $m_{PT,k}^* \leq m_{T,k}^* \leq m_{L,k}^*$. In fact, these might be equal, but for $k \geq 3$ no one knows. (For $k=1$ or 2, they are equal.)

Though it has not been proven, it seems likely to me that this bound is tight. That is, it is likely that

$$m_k^* = m_{PT,k}^*$$
• k = 3: The best known periodic tessellation in three dimensions is the lattice generated by the truncated octahedron, two of which are illustrated below. It has been proven that this is the best lattice. But it is not been proven that this is the best tessellation or periodic tessellation. However, it seems likely that it is.

The nmi of the truncated octahedron is \( m = 0.078543 \). The nmi of a 3-dimensional sphere is 0.0770. Therefore,

\[
0.0770 \leq m_3^* \leq 0.0785
\]

It is interesting that the basic group of the Weaire-Phelan partition (shown earlier) has nmi 0.078735 which is very nearly as small as that of the truncated octahedron.

---


---

**Properties**\(^10\) of \( m_{T,k}^* \)

• It is not known if \( m_{k+1}^* \leq m_k^* \) for all \( k \).

There is no known proof nor counterexample.

• Though \( m_k^* \) might not be decreasing with \( k \), if as I think is likely, it is true that \( m_k^* = m_{PT,k}^* \), then \( m_k^* \) can be shown to have a kind of “decreasing trend” called “subadditivity”, meaning that for any \( k,l \)

\[
m_{k+l}^* \leq \frac{k}{k+l} m_k^* + \frac{l}{k+l} m_l^*
\]

which implies (with appropriate use of the above)

\[
m_1^* \geq m_k^* \geq m_{2k}^*
\]

It also can be shown that subadditivity implies

\[
m_\infty^* = \lim_{k \to \infty} m_k^* = \inf m_k^*
\]

• A proof that subadditivity implies \( \lim = \inf \) can be found in Gallager's information theory book, Lemma 2, pp. 112,113.

The proof of subadditivity depends on the next fact:

---

\(^10\)For completeness, we include some of those that have already been mentioned.
• **Fact:** If \( S \subset \mathbb{R}^k \) and \( T \subset \mathbb{R}^l \), then

\[
M(S \times T) = \text{vol}(S) M(T) + \text{vol}(T) M(S)
\]

where \( M(S) = \int_S \|x\|^2 \, dx = MI \)

• **Proof of Fact:**

\[
M(S \times T) = \int_{S \times T} \|x\|^2 \, dx = \int_S \int_T (\|x\|^2 + \|y\|^2) \, dx \, dy
\]

\[
= \int_S (M(T) + \|y\|^2 \text{vol}(T)) \, dy = M(T) \text{vol}(S) + M(S) \text{vol}(T)
\]

**Partial proof of Subadditivity:**

This proof assumes \( m_k^* = m_{k^*} \). Let \( S \) and \( T \) be tessellating polyhedra with unit volumes that achieve \( m_k^* \) and \( m_l^* \), respectively. Then \( S \times T \) is also a tessellating polyhedron. And \( \text{vol}(S \times T) = 1 \). Therefore, applying the Fact,

\[
m_k^* \leq m(S \times T) = \frac{M(S \times T)}{(k+1)\text{vol}(S \times T)^{(k+l+2)/(k+l)}}
\]

\[
= \frac{M(T)\text{vol}(S) + M(S)\text{vol}(T)}{k+l} = \frac{1}{k+l} (M(T) + M(S))
\]

\[
= \frac{1}{k+l} (k \text{vol}(T) + (M(S) + )) = \frac{k}{k+l} m_k^* + \frac{1}{k+l} m_l^*
\]

I believe that a proof of subadditivity of this sort could be written assuming the weaker conjecture that the smallest inertial profile corresponds to a periodic tessellation, but I haven’t actually tried it.

• **Sphere Lower bound:**

\[
m_k^* \geq m(k\text{-dim sphere}) = \left(\frac{V_k}{k+2}\right)^{2/k} \to \frac{1}{2\pi e} = .0585 \equiv \frac{1}{17} \to \frac{1}{2\pi e}
\]

where \( V_k = \text{vol. of k-dim. sphere w radius 1} \)

\[
m_k^* \to \frac{1}{2\pi e} = .0585 \equiv \frac{1}{17} \text{ as } k \to \infty.
\]

Proved by Zamir and Feder\textsuperscript{11}.

• **Upper bounds:** \( m_k^* \leq m(S) \) for best known tessellation or periodic tessellation.

Such bounds may continue to improve as people learn of better tessellations.

• There is a conjectured lower bound in Conway and Sloane’s book, p. 59-62. It is tighter than the sphere lower bound

• Summary: \( m_k^* \) decreases with \( k \) (though not necessarily monotonically) from \( 1/12 = .0833 \) at \( k = 1 \) to \( m_\infty^* = 1/2\pi e = .0585 \approx 1/17 \), which translates to a potential SNR gain of 1.53 dB.

• The book by Conway and Sloane has a summary of what is known about the best tesselating polytopes. Sloane also has a website that may contain further updates.

• We could use a better name for \( m_k^* \). Any suggestions?

### The Best Known Tesselating Polytopes

<table>
<thead>
<tr>
<th>dimension</th>
<th>polytope</th>
<th>( m_k^* )</th>
<th>best known, ( \text{conj'd lower bound} )</th>
<th>sphere lower bound</th>
<th>gain (dB) 10 ( \log \frac{m_k^<em>}{m_k^</em>} )</th>
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<td>.0833</td>
<td>.0833</td>
<td>0</td>
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<td>.0766, .0796'</td>
<td>.0770</td>
<td>.16</td>
</tr>
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<td>unknown</td>
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<td>.0747'</td>
<td>.0775</td>
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<td>.0585</td>
<td>.0585</td>
<td>1.53</td>
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Properties of the Zador Factor $\beta_k$

(1) If $Y = aX + b$, where $a \neq 0$, then $\beta_{Y,k} = \beta_{X,k}$. This shows $\beta_k$ depends on the shape of the density, and is invariant to scaling or shifting.

Derivation: This can be derived directly, or from Property (2) with $A$ being a diagonal matrix with $a$'s on the diagonal and $|A| = a^k$, and with $\sigma_Y^2 = a^2 \sigma_X^2$. It could also be derived from the fact that scaling and translating scales the OPTA function by the factor $a^2$, and since the OPTA function equals Zador's formula, it must also scale the later. As a result it must scale $\beta$. 

SNR gains due to decreasing $M^k$

1.53dB limit

1 10 100 1000
dimension

upper bound based on C&S conjecture

lower bound based on known lattice tessellations

known values of $M^k$
(2) If $Y = AX + b$, where $A$ is a nonsingular square matrix, then
\[ \beta_{Y,k} = \frac{\sigma_X^2}{\sigma_Y^2} |A|^{2/k} \beta_{X,k} \]

Note that $\beta$ is not affected by the addition of the constant $b$.

Derivation: We have $f_Y(y) = |A|^{-1} f_X(A^{-1}(y-b))$. Therefore,
\[
\beta_{Y,k} = \frac{1}{\sigma_Y^2} \left( \int f_Y(y)^{k/k+2} dy \right)^{(k+2)/k} = \frac{1}{\sigma_Y^2} \left( \int (|A|^{-1} f_X(A^{-1}(y-b)))^{k/k+2} dy \right)^{(k+2)/k}
\]
\[
= \frac{1}{\sigma_Y^2} |A|^{-1} \left( \int f_X(x)^{k/k+2} |A|^2 dx \right)^{(k+2)/k}, \text{ with } x = A^{-1}(y-b), y = Ax + b, dy = |A| dx
\]
\[
= \frac{1}{\sigma_Y^2} |A|^{2/k} \left( \int f_X(x)^{k/k+2} dx \right)^{(k+2)/k} = \frac{1}{\sigma_Y^2} |A|^{2/k} \sigma_X^2 \beta_{X,k}
\]

(3) If $Y = AX + b$ and $A$ is a $k \times k$ orthogonal matrix (i.e. $A^{-1} = A^\dagger$), then $\beta_{Y,k} = \beta_{X,k}$.

It should be intuitive that the opta for $Y$ is the same as for $X$, because one can rotate and translate an optimal VQ for $X$ to get a VQ with the same performance for $Y$, and vice versa.

Derivation: For an orthogonal matrix $\|Ax\| = \|x\|$ for all $x$. Therefore,
\[ \sigma_Y^2 = \frac{1}{k} E\|Y-EY\|^2 = \frac{1}{k} E\|A(X-EX)\|^2 = \frac{1}{k} E\|X-EX\|^2 = \sigma_X^2 \]

Next $|A| = \prod_{i=1}^k \lambda_i = 1$, where the $\lambda_i$'s are the eigenvalues, which all have magnitude one, because $Ax = \lambda x$ implies $\|x\| = \|Ax\| = \|\lambda x\| \Rightarrow |\lambda| = 1$.

(4) If $Y = AX + b$, where $A$ is a $k \times k$ diagonal matrix with diagonal elements $a_1, \ldots, a_k$, then
\[
\beta_{Y,k} = \left( \prod_{i=1}^k a_i^2 \right)^{1/k} \sum_{i=1}^k \frac{\sigma_{X,i}^2}{\sum_{i=1}^k a_i^2 \sigma_{X,i}^2} \beta_{X,k}
\]
\[
= \left( \prod_{i=1}^k a_i^2 \right)^{1/k} \frac{1}{\frac{1}{k} \sum_{i=1}^k a_i^2} \beta_{X,k} \text{ if the } \sigma_{X,i}^2 \text{ 's are all the same}
\]

Zador-37

Zador-38
(5) If $X_1, \ldots, X_k$ are independent, then
\[
\beta_k = \frac{1}{\sigma_X^2} \prod_{i=1}^{k} \left( \int f_i(x)^{k/(k+2)} \, dx \right)^{(k+2)/k}
\]
Derivation:
\[
\beta_{X,k} = \frac{1}{\sigma_X^2} \left( \int f_1(x_1)^{k/(k+2)} \cdots f_k(x_k)^{k/(k+2)} \, dx \right)^{(k+2)/k}
\]
\[
= \frac{1}{\sigma_X^2} \left( \int f_1(x_1)^{k/(k+2)} \, dx \cdots \int f_k(x_k)^{k/(k+2)} \, dx \right)^{(k+2)/k}
\]
(6) If $X_1, \ldots, X_k$ are independent and identical (IID) with variance $\sigma^2$
\[
\beta_{X,k} = \frac{1}{\sigma^2} \left( \int f_1(x)^{k/(k+2)} \, dx \right)^{k+2}
\]
(7) If $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_k$ have the same marginal distributions, but $Y_1, \ldots, Y_k$ are independent, then
\[
\beta_{X,k} \leq \beta_{Y,k}
\]
with equality iff the $X_i$'s are independent.
This illustrates how dependence among the $X_i$'s (equivalently memory in the source) reduces the value of $\beta_k$.

(8) Suppose $X_1, \ldots, X_k$ are Gaussian
(a) Independent Gaussian case
\[
\beta_{X,k} = 2\pi \left( \frac{k+2}{k} \right)^{(k+2)/2} \left( \prod_{i=1}^{k} \sigma_i^2 \right)^{1/k}
\]
Derivation: From (5)
\[
\beta_{X,k} = \frac{1}{\sigma_X^2} \prod_{i=1}^{k} \left( \int \left( \frac{1}{\sqrt{2\pi\sigma_i^2}} \right)^{k/(k+2)} \, dx \right)^{(k+2)/k}
\]
\[
= \frac{1}{\sigma_X^2} \prod_{i=1}^{k} \left( \int \left( \frac{1}{\sqrt{2\pi\sigma_i^2}} \right)^{k/(k+2)} \, dx \right)^{(k+2)/k}
\]
\[
= \frac{1}{\sigma_X^2} \prod_{i=1}^{k} \left( \frac{2\pi\sigma_i^2}{(2\pi\sigma_i^2)^{1/(2(k+2))}} \right)^{(k+2)/k}
\]
\[
= 2\pi \left( \frac{k+2}{k} \right)^{(k+2)/2} \left( \prod_{i=1}^{k} \sigma_i^2 \right)^{1/k}
\]
(b) IID Gaussian case

\[ \beta_k = 2\pi \left( \frac{k+2}{k} \right)^{(k+2)/2} \downarrow \beta_\infty = 2\pi e = 17.1 \quad \text{as} \quad k \to \infty \]

It can be shown that these \( \beta_k \)'s decrease monotonically up to \( \beta_\infty \).

The fact that the \( \beta_k \)'s decrease with \( k \) demonstrates that even for independent random variables there are gains to increasing the dimension of vector quantization beyond those due to the reductions in \( m_k^* \).

Note:

\[ \left( \frac{k+2}{k} \right)^{(k+2)/2} = \exp\left\{ \frac{k+2}{2} \ln \frac{k+2}{k} \right\} = \exp\left\{ \frac{k+2}{2} \ln (1+ \frac{2}{k}) \right\} \]

\[ \approx \exp\left\{ \frac{k+2}{2} \frac{2}{k} \right\} = \exp\left\{ \frac{k+2}{k} \right\} \to e \]

(c) For a stationary, Gaussian first-order autoregressive source with correlation coefficient \( \rho \), (i.e. \( X \) can be modelled as \( X_k = \rho X_{k-1} + W_k \), where \( W_k \)'s are IID Gaussian and independent of past \( X_j \)'s.)

\[ \beta_k = 2\pi \left( \frac{k+2}{k} \right)^{(k+2)/2} (1-\rho^2)^{(k-1)/k} \downarrow \beta_\infty = 2\pi e (1-\rho^2) \]

(d) Correlated Gaussian random vector with covariance matrix \( K \)

\[ \beta_k = 2\pi \left( \frac{k+2}{k} \right)^{(k+2)/2} \left| K \right|^{1/k} \sigma_X^{-2} \]

Derivation: We assume \( E X = 0 \) because the mean does not affect \( \beta \).
We find \( \beta_k \) by transforming \( X \) to an independent vector \( U \) via an orthogonal transform. From (3), \( \beta_k = \beta_{U,k} \); \( \beta_{U,k} \) can be found from (a) above.

Accordingly, let \( U = AX \), where \( A \) is the Karhunen-Loeve transform, i.e. its rows \( z_1, \ldots, z_k \) are an orthonormal set of eigenvectors for \( K \). Let \( \lambda_1, \ldots, \lambda_k \) be the corresponding eigenvalues. Then

\[ K_U = E U U^t = E AX X^t A^t = A E X X^t A^t = A K_X A^t \]

\[ = A \left[ \begin{array}{c} \lambda_1 \ z_1 \\ \vdots \\ \lambda_k \ z_k \end{array} \right] = \left[ \begin{array}{c} \lambda_1 \\ \vdots \\ \lambda_k \end{array} \right] \]

from which we see that \( U \) is uncorrelated and, also, independent since it is Gaussian. Using the fact that \( A \) is orthonormal and (a) above, we have

\[ \beta_{X,k} = \beta_{U,k} = 2\pi \left( \frac{k+2}{k} \right)^{(k+2)/2} \left( \frac{\prod_{i=1}^{k} \sigma_i^2}{\sigma_X^2} \right)^{1/k} \]

\[ = 2\pi \left( \frac{k+2}{k} \right)^{(k+2)/2} \frac{\left| K \right|^{1/k}}{\sigma^2} \]

(9) Uniform density
(a) Independent (each $X_i$ uniform on some interval)
\[ \beta_k = 12 \left( \prod_{i=1}^{k} \sigma_i^2 \right)^{1/k} \]
(b) IID uniform on an interval
\[ \beta_k = 12 \]
Notice that it is the same for all $k$.
(c) Uniform on an arbitrary $k$-dimensional set $B$
\[ \beta_k = \frac{\text{vol}(B)^{2/k}}{\sigma^2} \]

(10) Laplacian density \( (f_X(x) = \frac{1}{\sqrt{2}} e^{-\sqrt{2}|x|}, \sigma^2 = 1) \)

(a) Independent
\[ \beta_k = \text{to be determined} \]
(b) IID
\[ \beta_k = 2 \left( \frac{k+2}{k} \right)^{k+2} \rightarrow \beta_\infty = 2 e^2 = 14.8 \text{ as } k \rightarrow \infty \]

Asymptotic Properties of Optimal Quantizers

Consider an optimal quantizer with large $M$ and point density, approximately, $\lambda^*_k$. Let $S_x$ denote the cell containing $x$.

- **Cell volume**
\[ |S_x| \equiv \frac{1}{M \lambda^*_k(x)} = \frac{c}{M f_X(x)^{k/(k+2)}} \]
Smaller where $f$ is larger, which is not surprising.

- **Cell probability**
\[ \Pr(S_x) \equiv f_X(x) |S_x| \equiv f_X(x) \frac{c}{M} f_X(x)^{k/(k+2)} \]
\[ = \frac{c}{M} f_X(x)^{2/(k+2)} \]
Larger where $f$ is larger
• Cell distortion
\[ \frac{1}{k} \int_{S_x} ||x-Q(x')||^2 \, f_{x}(x') \, dx' = \frac{1}{k} \int_{S_x} ||x-Q(x')||^2 \, dx' \]
\[ = \frac{1}{k} f_{x}(x) \, M(S_{x}) = \frac{1}{k} \, f_{x}(x) \, k \cdot m_{k} \cdot |S_{x}|^{(k+2)/k} \]
\[ = f_{x}(x) \, m_{k} \cdot \left( \frac{c}{M} \right)^{(k+2)/k} \]
\[ = m_{k} \cdot \left( \frac{c}{M} \right)^{(k+2)/k} \]

Same for all x; i.e. all cells contribute the same to the distortion.

• Conditional cell distortion
\[ \frac{1}{k} \int_{S_x} ||x-Q(x')||^2 \, f_{x}(x'|x \in S_{x}) \, dx' = \frac{1}{k} \int_{S_x} ||x-Q(x')||^2 \, \frac{f_{x}(x')}{Pr(S_{x})} \, dx' = \frac{\text{cell distortion}}{Pr(S_{x})} \]
\[ = \frac{M}{c} f_{x}(x)^{-2/(k+2)} \, m_{k} \left( \frac{c}{M} \right)^{(k+2)/k} \text{ using two previous bullets} \]
\[ = f_{x}(x)^{-2/(k+2)} \, m_{k} \left( \frac{c}{M} \right)^{2/k} \]

Inversely proportional to cell probability. Smaller where f is larger.

The OPTA Function for Stationary Sources

For a stationary source, we can use a quantizer of any dimension. Thus, it is interesting to know what is the least possible distortion of any quantizer with rate R or less and any dimension. That is, we seek the "ultimate" OPTA function:\[ \delta(R) = \inf_{\text{VQ's with rate } R \text{ or less}} D(\text{VQ}) \]
\[ = \inf_{k} \delta(k,R) \]

Equivalently,
\[ S(R) = \sup_{\text{VQ's with rate } R \text{ or less}} \text{SNR}(\text{VQ}) \]
\[ = \sup_{k} S(k,R) \]

There is no VQ that achieves \( \delta(R) \) exactly and no value of k such that \( \delta(k,R) = \delta(R) \). However, by definition of an "inf", for any R and any small tolerance \( \epsilon > 0 \), there is a VQ with rate R or less and distortion \( D \leq \delta(R) + \epsilon \), and there is a value of k such that \( \delta(k,R) \leq \delta(R) + \epsilon \). That is, one can come arbitrarily close to achieving the inf.

\[ ^{12} \text{Note that this is a function of } R, \text{ not } M. \text{ It would not be so interesting to make this a function of } M, \text{ because quantizers with the same } M \text{ but different dimensions have different rates.} \]
An "inf" is like a "min" except it works in cases where "min" does not.

Example,
\[
\min_{x \in (0,1)} x^2 \text{ does not exist}
\]

Defn: \( \inf_{x \in G} f(x) = \text{largest number } y \text{ such that } y \leq f(x) \text{ for all } x \in G \)

Example,
\[
\inf_{x \in (0,1)} x^2 = 0
\]

because \( 0 \leq x^2 \) for all \( x \in (0,1) \) and there is no larger number \( y \) such that \( y \leq x^2 \) for all \( x \in (0,1) \)

Defn: \( \sup_{x \in G} f(x) = \text{smallest number } y \text{ such that } y \geq f(x) \text{ for all } x \in G \)

- "inf" and "sup" are short for "infimum" and "supremum"
- covered in Math 451

Properties of the OPTA functions of Stationary Sources

The OPTA's have a decreasing trend as \( k \) increases. However, it is not known if \( \delta(k,R) \geq \delta(k+1,R) \) for all \( k \). All that is known is:

Fact: The OPTA function \( \delta(k,R) \) is subadditive in \( k \); i.e.
\[
\delta(k+l,R) \leq \frac{k}{k+l} \delta(k,R) + \frac{l}{k+l} \delta(l,R) \text{ for any } k,l
\]

Proof: Let \( Q \) = product of \( Q_k \) and \( Q_l \), which are \( k \) and \( l \) dim'l VQ's with rate \( R \) or less, and with \( D(Q_k) \equiv \delta(k,R) \) and \( D(Q_l) \equiv \delta(l,R) \). Then
\[
R(Q) = \frac{k}{k+l} R(Q_k) + \frac{l}{k+l} R(Q_l) \leq R
\]
\[
D(Q) = \frac{k}{k+l} D(Q_k) + \frac{l}{k+l} D(Q_l) \equiv \frac{k}{k+l} \delta(k,R) + \frac{l}{k+l} \delta(l,R)
\]
Therefore,
\[
\delta(k,R) \leq D(Q) \equiv \frac{k}{k+l} \delta(k,R) + \frac{l}{k+l} \delta(l,R)
\]

Fact: Subadditivity implies
\[
\delta(1,R) \geq \delta(k,R) \geq \delta(mk,R) \text{ for any } m,k
\]
\[
\delta(R) = \inf_k \delta(k,R) = \lim_{k \to \infty} \delta(k,R)
\]

Proof: For any subadditive sequence, \( \lim = \inf \) (cf. Gallager, p. 112, 113)
Other Properties of the OPTA Function

For a stationary source with continuous random variables and fixed-rate VQ:

- \( \delta(0) = \sigma^2 \) (recall that \( \delta(k,0) = \sigma^2 \))
- \( \delta(R) \) decreases monotonically toward zero as \( R \to \infty \).
- \( \delta(R) \) is a continuous, convex cup function of \( R \).

This is not like \( \delta(k,R) \), which has a stair-step form.

Sketch of proof of convexity: Given target rates \( R_1, R_2 \) and \( \alpha, \ 0 < \alpha < 1 \), we must show

\[
\delta(\alpha R_1 + (1-\alpha)R_2) \leq \alpha \delta(R_1) + (1-\alpha) \delta(R_2).
\]

First, suppose \( \alpha = 1/2 \). Let \( Q_1, Q_2 \) be VQ’s with large dimension \( k \), with rates at most \( R_1 \) and \( R_2 \) and distortions \( D_1 \equiv \delta(R_1) \) and \( D_2 \equiv \delta(R_2) \), respectively. \((Q_1, Q_2 \) exist by the df of the OPTA function.)

Consider another VQ, denoted \( Q \), with dimension \( 2k \) created by using \( Q_1 \) followed by \( Q_2 \). \((Q \) time shares between \( Q_1 \) and \( Q_2 \).)

Then

\[
R(Q) = \frac{1}{2} (R_1 + R_2)
\]

\[
D(Q) = \frac{1}{2} (D_1 + D_2).
\]

Since \( \delta(\frac{1}{2}R_1 + \frac{1}{2}R_2) \) is least dist'n of any quant, with rate \( \frac{1}{2}R_1 + \frac{1}{2}R_2 \),

\[
\delta(\frac{1}{2}R_1 + \frac{1}{2}R_2) \leq D(Q) = \frac{1}{2} (D_1 + D_2) \equiv \frac{1}{2} (\delta(R_1) + \delta(R_2))
\]

A somewhat sharper argument can demonstrate the above without the "\( \equiv \)". It could also use a time sharing that applies to any value of \( \alpha \), thereby establishing convexity.

Proof of Continuity: Convexity implies continuity, except possibly at \( R = 0 \). But it can be shown that \( \delta(R) \) is continuous at \( R = 0 \), too.
Recall that for large $R$
\[
\delta(k,R) \equiv Z(k,R) \overset{\Delta}{=} m_k \beta_k \sigma^2 2^{-2R}
\]

What happens as $k$ increases?

Recall: $m_k$'s have a decreasing trend. We believe they are subadditive.

How about the $\beta_k$'s? They, too, have a decreasing trend. It is not known if $\beta_{k+1} \leq \beta_k$ for all $k$ and all sources, however:

**Fact:** For a stationary source, $\beta_k$ is submultiplicative; i.e.
\[
\beta_{k+1} \leq \left(\beta_k \beta_l\right)^{1/(k+l)}
\]
equivalently $\log \beta_k$ is subbadditive:
\[
\log \beta_{k+1} \leq \frac{k}{k+l} \log \beta_k + \frac{l}{k+l} \log \beta_l
\]

From submultiplicativity it follows that
\[
\log \beta_1 \geq \log \beta_k \geq \log \beta_{mk}
\]
for every $m, k \geq 1$

and also that \(\lim_{k \to \infty} \log \beta_k = \inf_k \log \beta_k\)

and consequently that
\[
\beta_\infty \overset{\Delta}{=} \lim_{k \to \infty} \beta_k = \inf_k \beta_k
\]

**Sketch of proof of Fact:** Let $f_k(x)$ denote the $k$-dimensional density of $X_1, \ldots, X_k$. The proof uses the

**Triple Holder inequality:**

If $f$, $g$ and $h$ are nonnegative functions, and $p$, $q$, $r$ are nonnegative numbers such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, then
\[
\int f(x) g(x) h(x) \, dx \leq \left( \int f^p(x) \, dx \right)^{1/p} \left( \int g^q(x) \, dx \right)^{1/q} \left( \int h^r(x) \, dx \right)^{1/r}
\]

We apply this inequality to
\[
\left( \sigma^2 \beta_{k+l} \right)^{(k+l)/2} = \int f_k^{(k+l)/2} (x,y) \, dx dy
\]

with the integrand factored as
\[
f_k^{(k+l)/2}(x,y) = f(x,y) \, g(x,y) \, h(x,y),
\]

where
\[
f(x,y) = \left( f_k(x,y) f_k^{2/(k+2)}(x) \right)^{k/(k+l+2)}, \quad g(x,y) = \left( f_k(x,y) f_l^{2/(l+2)}(y) \right)^{(l+2)/2}\]
\[
h(x,y) = \left( f_k^{k/(k+2)}(x) f_l^{l/(l+2)}(y) \right)^{2/(k+l+2)}
\]

We let $p = (k+l+2)/k$, $q = (k+l+2)/l$ and $r = (k+l+2)/2$.

After simplifying, we find
\[
\left( \sigma^2 \beta_{k+l} \right)^{(k+l)/2} \leq \left( \sigma^2 \beta_k \right)^{k/(k+l+2)} \left( \sigma^2 \beta_l \right)^{l/(k+l+2)}
\]

which implies the desired result.
The decreasing trends for $m_k^*$ and $\beta_k$ indicate that one can't do better than to choose $k$ large. Indeed, we need $k$ to be large in order to approach the best possible performance. The following theorem summarizes.

**Theorem:** For a stationary source and large values of $R$,

$$\delta(R) \equiv Z(R) \stackrel{\Delta}{=} \frac{1}{2\pi e} \sigma^2 \beta_\infty 2^{2R}$$

Equivalently,

$$S(R) \equiv 10 \log_{10} \frac{\sigma^2}{Z(R)} = 6.02 R - 10 \log_{10} \frac{1}{2\pi e \beta_\infty}$$

We again see the 6 dB gain per bit.

**Relationship of kth-order OPTA to overall OPTA**

$$S(k,R) = 6.02 R - 10 \log_{10} m_k^* \beta_k$$

$$= 6.02 R - 10 \log_{10} m_\infty^* \beta_\infty - 10 \log_{10} \frac{m_k^*}{m_\infty} - 10 \log_{10} \frac{\beta_k}{\beta_\infty}$$

$$= S(R) - 10 \log_{10} \frac{m_k^*}{m_\infty} - 10 \log_{10} \frac{\beta_k}{\beta_\infty}$$

From this we see explicitly how $S(k,R)$ improves with $k$ through decreases in $m_k$ and $\beta_k$.

**Gauss-Markov Source, $\rho = .9$**

$$S(k,R) = S(R) - 10 \log_{10} \frac{m_k^*}{m_\infty} - 10 \log_{10} \frac{\beta_k}{\beta_\infty}, \quad R = 3$$
Question:

Why does $\beta_k$ get better as $k$ increases, even for an IID source?

We'll consider this question after discussing some properties of $\beta_\infty$. 

IID Gaussian source

\[ S(k,R) = S(R) - 10 \log_{10} \frac{m_k^*}{m_\infty} - 10 \log_{10} \frac{\beta_k}{\beta_\infty}, \quad R = 3 \]
Properties of $\beta_\infty$

(1) For a Gaussian source, recall

$$\beta_k = 2\pi \left(\frac{k+2}{k}\right)^{(k+2)/2} \frac{|K^{(k)}|^{1/k}}{\sigma^2}$$

where $K^{(k)}$ is the $k \times k$ by covariance matrix of $(X_1, \ldots, X_k)$. Recall that

$$2\pi \left(\frac{k+2}{k}\right)^{(k+2)/2} \downarrow 2\pi e \text{ as } k \to \infty$$

For a stationary Gaussian source

$$|K^{(k+1)}|^{1/(k+1)} \leq |K^{(k)}|^{1/k} \quad \text{(this will be shown later in the course)}$$

$$|K^{(k)}|^{1/k} \downarrow Q \text{ as } k \to \infty \quad \text{(will also be shown later)}$$

where

$$Q \triangleq \exp\left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln S_X(\omega) \, d\omega\right\}$$

$$= \text{MSE of optimum linear predictor for } X \text{ based on } X_{i-1}, X_{i-2}, \ldots$$

$$= \text{"one-step prediction error"}$$

and $S_X(\omega) = \text{power spectral density of } X$.

It follows that

$$\beta_k \downarrow \beta_\infty = \frac{2\pi e Q}{\sigma^2} \text{ as } k \to \infty$$

(2) Upper bound for an arbitrary stationary source

$$\beta_\infty \leq \frac{2\pi e Q}{\sigma^2} \quad \text{with equality iff } X \text{ is Gaussian}$$

This shows that Gaussian sources have the largest $Z(R)$ among sources with a given power spectral density or autocorrelation function. In other words, Gaussian sources are the hardest to quantize.

Note, for example, that even though an IID Laplacian source has a larger $\beta_1$ it has a smaller $\beta_\infty$.

Proof: Postponed to variable-rate VQ discussion, for reasons that will be clear then.

(3) $\beta_k$ tends to be smaller and to decrease more with $k$ for sources with memory than for memoryless sources. (This is an admittedly rough rule of thumb.)

(4) For a stationary, Gaussian first-order autoregressive source with correlation coefficient $\rho$, $Q = 1-\rho^2$ and

$$\beta_k = 2\pi \left(\frac{k+2}{k}\right)^{(k+2)/2} (1-\rho^2)^{(k-1)/k} \to \beta_\infty = 2\pi e (1-\rho^2)$$
Why Fixed-Rate VQ Outperforms Fixed-Rate SQ

We want to understand what specific characteristics of vector quantization improve with dimension, and by how much.

We will compare a fixed-rate k-dimensional VQ, denoted $Q_k$, to a fixed-rate scalar quantizer $Q_1$, both having rate $R$.

To make it fair, we compare characteristics (point density and inertial profile) of $Q_k$ to those of the k-dimen'l "product" VQ, denoted $Q_{pr,k}$, formed by using $Q_1$ $k$ times in succession.

The "product" codebook contains all $k$-tuples formed from scalar quant. levels.

In the "product" partition, each cell is the Cartesian product of the scalar cells corresponding to the components of its codevector.

The "product" quantization rule is

$$Q_{pr,k}(x) = (Q_1(x_1), Q_1(x_2), \ldots, Q_1(x_k))$$

As we know

$$R(Q_{pr,k}) = R(Q_1) = R$$

$$D(Q_{pr,k}) = D(Q_1) = \delta(1,R)$$

Let $\lambda_1$ be the point density of the scalar quantizer.

The cell $S_x$ of the product quantizer containing $x$ is a rectangle:

$$S_x = \frac{1}{2^R\lambda_1(x_1)} \times \frac{1}{2^R\lambda_1(x_2)} \times \frac{1}{2^R\lambda_1(x_3)} \times \ldots \times \frac{1}{2^R\lambda_1(x_k)}$$

with volume

$$|S_x| = \frac{1}{2^{kR}} \times \frac{1}{\lambda_1(x_1)\lambda_1(x_2)\ldots\lambda_1(x_k)}$$

The point density of the product quantizer is

$$\lambda_{pr,k}(x) = \frac{1}{2^{kR}|S_x|} = \lambda_1(x_1)\lambda_1(x_2)\ldots\lambda_1(x_k) \quad \text{(It's a product!)}$$

The inertial profile of the product quantizer is

$$m_{pr,k}(x) = \frac{1}{12} \left[ \frac{1}{k} \sum_{i=1}^{k} \frac{1}{\lambda_1(x_i)^2} \right]^{1/k} \geq m_1 \cdot \left[ \frac{1}{k} \sum_{i=1}^{k} \frac{1}{\lambda_1(x_i)^2} \right]^{1/k}$$

Notice that both the point density and inertial profile of the product quantizer are determined by the point density of the scalar quantizer.
Now consider the ratio of the distortion of the scalar quantizer (i.e. the product quantizer) to that of the VQ. We consider this to be a "loss" \( L \):

\[
L = \frac{D(Q_1)}{D(Q_k)} = \frac{D(Q_{pr,k})}{D(Q_k)}
\]

Let us apply Bennett's integral to both terms using the notation

\[
B(k, m, \lambda, f) = \int m(x) \lambda^{2/k}(x) f(x) \, dx = \text{Bennett's integral}
\]

With this notation, the loss is

\[
L = \frac{D(Q_1)}{D(Q_k)} = \frac{D(Q_{pr,k})}{D(Q_k)} \approx \frac{B(k, m_{pr,k}, \lambda_{pr,k}, f)}{B(k, m_k, \lambda_k, f)}
\]

Now let us factor \( L \) into two terms, reflecting the shortcomings of the product inertial profile and product point density, respectively:

\[
L = \frac{B(k, m_{pr,k}, \lambda_{pr,k}, f)}{B(k, m_k, \lambda_{pr,k}, f)} \times \frac{B(k, m_k, \lambda_{pr,k}, f)}{B(k, m_k, \lambda_k, f)}
\]

\[
= \text{cell shape loss} \times \text{point density loss}
\]

\[
= L_{ce} \times L_{pt}
\]

\( L_{ce} \) is the factor by which distortion increases due to the inertial profile being \( m_{pr,k} \) instead of the more desirable \( m_k \).

\( L_{pt} \) is the factor by which distortion increases due to the point density being \( \lambda_{pr,k} \) instead of the more desirable \( \lambda_k \).

Now assume \( k \) is large and the \( k \)-dimensional VQ is optimal, so that

\[
m_k(x) = m_k^* \equiv m_\infty = \frac{1}{2\pi e} \quad \text{and} \quad \lambda_k(x) = \lambda_k^* = c_k f^{k/(k+2)}(x) \equiv f(x)
\]

Then the loss of the scalar quantizer relative to the optimal high dim'l VQ becomes

\[
L = \frac{m_1^*}{m_\infty} \times \frac{\int 1 \sum_{i=1}^{k} \frac{1}{\lambda_1^2(x_i)} f(x) \, dx}{\int (\prod_{i=1}^{k} \lambda_1^2(x_i))^{1/k} f(x) \, dx} \times \frac{B(k, m_k^*, \lambda_{pr,k}, f)}{B(k, m_k, \lambda_k, f)}
\]

\[
= \text{cubic loss} \times \text{oblongitis loss} \times \text{point density loss}
\]

\[
= L_{cu} \times L_{ob} \times L_{pt}
\]

where we have factored the cell shape loss \( L_{ce} \) into the product of a "cubic loss" \( L_{cu} \) and an "oblongitis loss".

The cubic loss is that ratio of nmi of a cube to that of a high-dimensional, nearly spherical cell.

The oblongitis loss reflects the inability of a product quantizer to produce all cubes. Instead it produces rectangles and \( L_{ob} \) is the loss due to this.
Now to get concrete numbers, assume the source is IID.

Consider the choice of $\lambda_1$ to minimize the loss, i.e. to minimize $L_{ob} \times L_{pt}$.
($L_{cu}$ is not influence by $\lambda_1$)

- Choosing $\lambda_1(x)$ to be a constant on the set where $f_1(x)$ is large makes $L_{ob} \equiv 1$, which is good. However, $L_{pt}$ becomes very large.

- On the other hand choosing $\lambda_1(x) = f_1(x)$ causes
  \[
  \lambda_{pr,k}(x) \equiv \prod_{i=1}^k f_1(x_i) = f(x) \equiv c f_k^{k/(k+2)}(x) = \lambda_k^*(x) \quad (k \text{ large } \Rightarrow k/(k+2) \equiv 1)
  \]
  so that $L_{pt} = 1$.

In other words scalar quantization can produce the optimal point density!

This fact is often overlooked, because for IID Gaussian case, the product quantizer looks like it has a "cubical" point density, when it actually has a spherical one.

Unfortunately, however, for this $\lambda_1$, $L_{ob} = \infty$. 

optimal 128 pt VQ for pair of indep. Gaussian sources

product quantizer formed by optimal 12 pt quantizer used twice
The point density that minimizes $L_{ob} \times L_{pt}$ is the compromise

$$\lambda_1(x) = c f_1(x)^{1/3}$$

This makes $\lambda_1(x)$ "flatter" (more uniform) in regions where $f_1(x)$ is large than the previous choice of $\lambda_1(x) = f(x)$. Therefore, the rectangular cells are more nearly cubical in the important region where $f_1(x)$ is large, so there is less oblongitis loss.

The oblongitis and point density losses are larger for Laplacian than for the Gaussian density, because the Laplacian's sharper peak at the origin and heavier tail means that a good scalar quantizer must be more nonuniform. This causes more oblongitis, which in turn causes more compromising of the optimal point density in order to reduce the oblongitis.

The total losses in the right hand column are the potential gains of high dimensional VQ over scalar quantization.
In summary, for an IID source the shortcomings of scalar quantization relative to high-dimensional VQ are

(a) The cubic loss $L_{cu} = 1.53$ dB, which is a measure of its inability to produce cells with smaller nmi than a cube.

(b) The lack of sufficient degrees of freedom to simultaneously attain good inertial profile and good point density.

Shortcomings of Optimal k-dimensional VQ

One can similarly decompose the loss of optimal k-dimensional VQ (e.g. k=2) relative to high-dimensional VQ into point density, oblongitis, and space-filling losses by comparing the characteristics of an optimal k-dimensional VQ to that of an optimal high-dimensional VQ. To make the comparison, one considers the point density and inertial profile of the product quantizer formed by using the k-dimensional VQ many times. The point density and oblongitis losses are then defined in the same way as before. The space-filling loss, which is

$$L_{sp} = \frac{m_k}{m_\infty},$$

generalizes the "cubic loss" we considered for k=1. It is called the "space-filling" loss because it represents the loss due to the product quantizers inability to fill space with cells whose NMI is as good as those that induce $m_k$. (For k=1 it's a "cubic loss", for k=2 it's a "hexagonal loss".

As k increases, one finds:

(a) The "space filling" loss decreases to 1 (0 dB).

(b) There are more degrees of freedom, so less compromise is needed between the k-dimensional point density that minimizes oblongitis and that which minimizes point density loss. Consequently, when optimized, the oblongitis & point density losses are smaller, and decrease to 1 (0 dB).
### IID Gaussian Source

\[
L_{ob} = \left(\frac{k+2}{k}\right)^{k/2} e^{-k/(k+2)}, \quad L_{pt} = \frac{k+2}{k} e^{-2/(k+2)}, \quad L_{pt} L_{ob} = L_{shape} = \frac{\beta_k}{\beta_{\infty}} = \frac{1}{e} \left(\frac{k+2}{k}\right)^{(k+2)/2}
\]

---

#### Table

<table>
<thead>
<tr>
<th>(k')</th>
<th>(L_{sp})</th>
<th>(L_{ob})</th>
<th>(L_{ce} = L_c L_{ob})</th>
<th>(L_{pt})</th>
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<td>point density</td>
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Memory Loss

For sources with memory, i.e. sources with dependent random variables (e.g. autoregressive), scalar and low-dimensional quantization suffer an additional loss, namely the inability to exploit or fully exploit the memory, i.e the dependence between source samples.

Both oblongitis and point density losses can be factored into two terms, one of which expresses the quantizers inability to fully exploit the source memory.

\[
L = \frac{1}{12} \times \frac{1}{2 \pi e} \times \frac{\int \frac{1}{k} \sum_{i=1}^{k} \frac{1}{\lambda_i^2(x_i)} f_k(x) \, dx}{\int \frac{1}{k} \prod_{i=1}^{k} \lambda_i^2(x_i) \prod_{i=1}^{k} \lambda_i^2(x_i) \frac{1}{k} \, dx} \times \frac{1}{\lambda_k^2} f_k(x) \, dx}
\]

\[
L = \text{space loss} \times \text{oblongitis loss} \times \text{point density loss}
\]

\[
= L_{sp} \times L_{ob} \times L_{pt}
\]