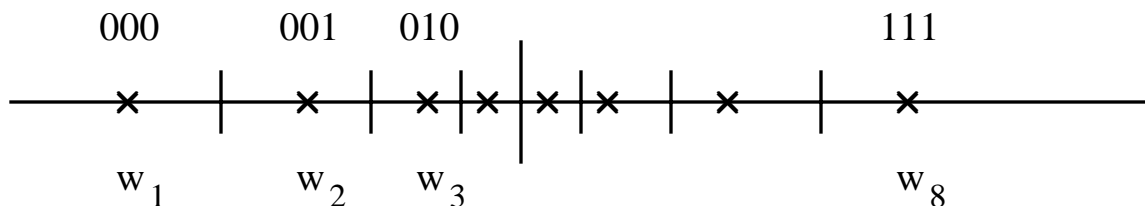


Nonuniform scalar quantizer

References: Sayood Chap. 9, Gersho and Gray, Chap.'s 5 and 6.

The basic idea: For a nonuniform source density, put smaller cells and levels where the density is larger, thereby making the quantization error smaller on the average.

Example: for source



Implementation: (think of M as large (e.g. 2^{16})) when thinking of complexity

1. Brute force:

Encoding by thresholding: Compare x to t_1, t_2, \dots , just as with USQ.

Complexity: $M-1$ comparisons, $M-1$ units of storage

Decoding by table lookup: Just as with USQ,

Store w_0, \dots, w_{M-1} in table. Given \underline{c}_I , decoder uses I to address table and to output $d(\underline{c}_I) = w_I$

Complexity: 0 ops/sample and M units of storage

2. Encoding by tree-structured thresholding:

Just as with USQ, successively compare x to test thresholds associated with the nodes of a binary tree. Show example.

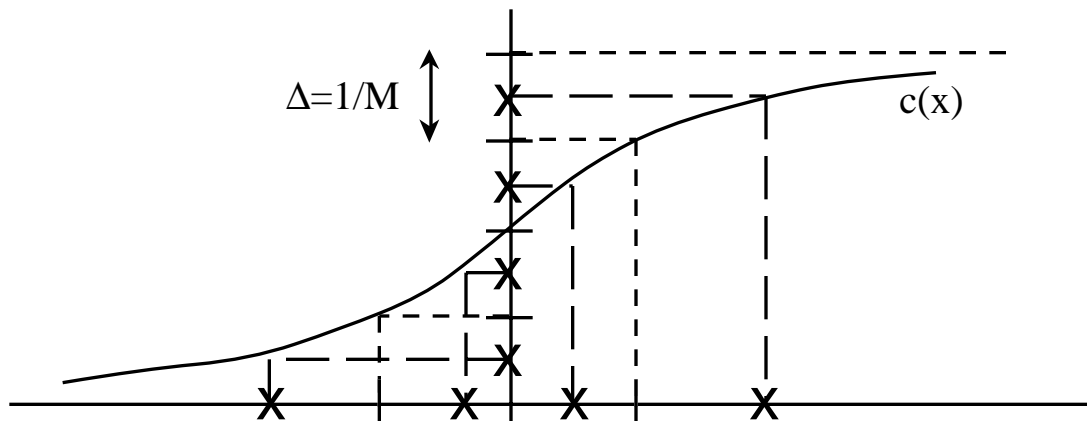
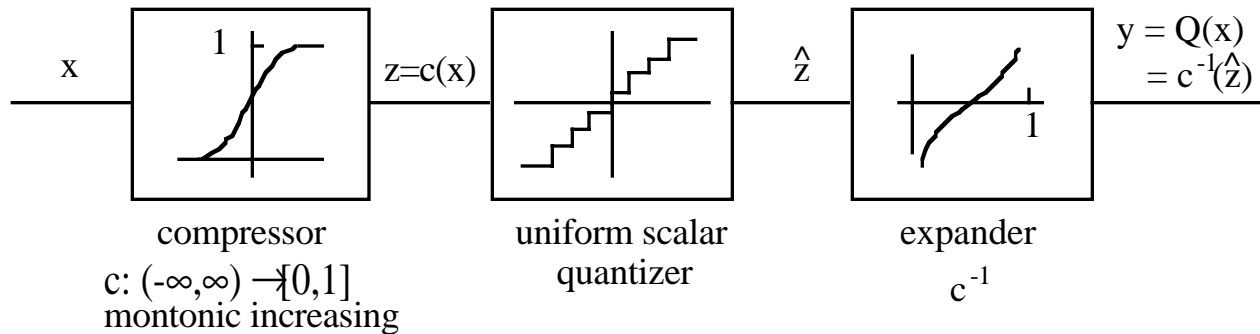
Complexity: $\log_2 M$ comparisons/sample. storage of $M-1$ test thresholds.

3. Companding:

encoder = compressor + uniform thresholds,

decoder = uniform levels + expander

The following shows the quantization rule Q .



Thresholds: $t_i = c^{-1}\left(\frac{i}{M}\right)$, $i = 1, \dots, M-1$;

Levels: $y_i = c^{-1}\left(\frac{i}{M} - \frac{1}{2M}\right)$, $i = 1, \dots, M$

One can implement any nonuniform quantizer with a compander.

Conversely, any compander is the implementation of some quantizer.

Complexity: that of USQ + compressor and expander functions

Examples:**A-law**

$$c(x) = \begin{cases} \frac{A|x|}{1+\ln A} \operatorname{sgn}(x), & |x| \leq \frac{x_{\max}}{A} \\ x_{\max} \frac{1+\ln(A|x|/x_{\max})}{1+\ln A} \operatorname{sgn}(x), & \frac{x_{\max}}{A} < |x| \end{cases}$$

Show picture:

European **PCM** standard: $A = 87.56$

 μ -law

$$c(x) = x_{\max} \frac{\ln(1+\mu|x|/x_{\max})}{\ln(1+\mu)}$$

North American **PCM** standard: $\mu = 255$

Things to do

1. **Design** quantizers to minimize MSE for given M and pdf
2. Find approximate expression for MSE of a quantizer.
3. Find **OPTA** function, $\delta_{sq}(R)$.
4. Find approximate expression for OPTA function.

Quantizer Design to minimize MSE

Given $f_X(x)$ and M, choose t_i 's and w_i 's to minimize MSE

1. Iterative approach.
2. Asymptotic approach.

Key Properties of Optimal Quantizers

Iterative design will be based on these.

1: For a given set of levels y_1, \dots, y_M , the thresholds that minimize MSE) are midway between levels, i.e.

$$t_i = \frac{y_i + y_{i+1}}{2}, \quad i = 1, \dots, M-1 \quad (*)$$

2: For a given set of thresholds t_1, \dots, t_{M-1} and source density $f_X(x)$, the levels that minimize MSE are the *centroids*

$$y_i = E[X | t_{i-1} < X < t_i] = \int_{-\infty}^{\infty} x f_X(x | t_{i-1} < X < t_i) du, \quad i = 1, \dots, M \quad (**)$$

where

$$f_X(x | t_{i-1} < X < t_i) = \begin{cases} \frac{f_X(x)}{\Pr(t_{i-1} < X < t_i)}, & t_{i-1} < x < t_i \\ 0, & \text{else} \end{cases}$$

Interpretation/derivation of (1):

The encoder controls the decoder, and so given x , the best thing is for the encoder to make the decoder produce the closest level to x . In other words, to minimize distortion for each x , we choose the thresholds so that $Q(x)$ is the level closest to x . Since this minimizes $(x-Q(x))^2$ for each x , it minimizes the distortion:

$$D = \int_{-\infty}^{\infty} (x-Q(x))^2 f(x) dx$$

It also implies the thresholds are halfway between the levels.

Can also derive (1) by equating derivative of distortion to zero, but why use a powerful mathematical tool when a simple observation suffices?

Exception to the rule: If the density is zero midway between two levels, then threshold can be anywhere in the interval containing the midpoint where the density is zero.

Interpretation/derivation of (2): decoder is MMSE estimator.

Recall estimation theory: If you must estimate the value of X based on having observed the occurrence of an event A , and if you want to choose the estimate \hat{x}_A of X to minimize

$$E[(X-\hat{x}_A)^2|A].$$

Then you should choose $\hat{x}_A = E[X|A]$.

(Recall: $E(X-c)^2$ is minimized by $c = EX$.)

Now,

$$\begin{aligned} D &= \sum_{i=1}^M E[(X-Q(X))^2|t_{i-1}<X<t_i] \Pr(t_{i-1}<X<t_i) \quad (\text{law of total expect'n}) \\ &= \sum_{i=1}^M E[(X-y_i)^2|t_{i-1}<X<t_i] \Pr(t_{i-1}<X<t_i) \end{aligned}$$

We minimize the sum by minimizing each term, which is done by choosing

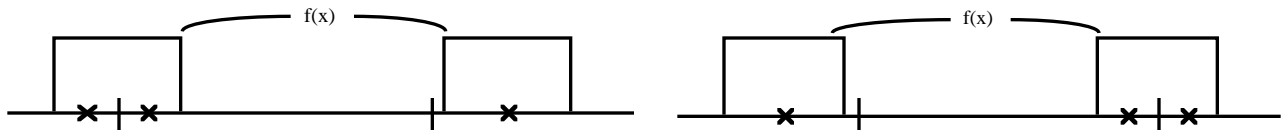
$$y_i = E[X|t_{i-1}<X<t_i]$$

Corollary to 1 and 2:

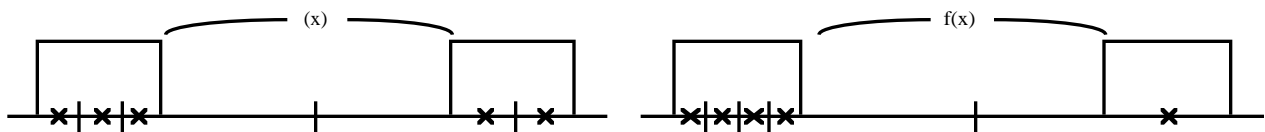
Given a size M and a source density $f_X(x)$, the best scalar quantizer (i.e. the one with smallest MSE) satisfies (*) and (**). Consequently, 1 and 2 are called optimality properties.

Notes:

- There may be more than one optimal quantizer. Example:



- Even if there is only one optimal quantizer, there may be more than one quantizer that satisfies the properties, in which case the best quantizer is one of them. In other words, the optimality properties are necessary for optimality, but not sufficient. Example of two quantizers that satisfy the optimality criteria.



And not all VQ's that satisfy (*) and (**) are best.

- If $\ln(f_X(x))$ is a strictly concave function of x , then there is only one optimal quantizer, there is only one quantizer with M levels that satisfies (*) and (**), and this is the unique best quantizer. Example: if f_X is Gaussian, then $\ln(f_X(x)) = -x^2/2\sigma^2 + \text{constant}$, which is strictly concave.

For a Laplacian density $\ln(f_X(x))$ is not strictly concave. However, it has been shown that for this special case, there is again only one quantizer with M levels that satisfies (*) and (**), and this is the unique best quantizer.

Reference: P.E. Fleischer, "Sufficient conditions for achieving minimum distortion in a quantizer," IEEE Int. Conv. Rec, Part I, p. 104, 1964.

Fact: A quantizer satisfies (*) and (**) iff it is a **local optimum**.

Defn: A quantizer is **locally optimum** if all sufficiently small perturbations increase or maintain distortion

What do we mean by "sufficiently small perturbation"? Consider replacing the sets of levels $\{y_1, \dots, y_M\}$ and thresholds by the sets $\{t_1 + \epsilon u_1, \dots, t_{M-1} + \epsilon u_{M-1}\}$ and $\{y_1 + \epsilon z_1, \dots, y_M + \epsilon z_M\}$ for some small ϵ and arbitrary sets $\{u_1, \dots, u_{M-1}\}$ and $\{z_1, \dots, z_M\}$. A quantizer is locally optimum iff for any such \underline{u} and \underline{z} , there is an ϵ_0 such that for all $\epsilon \leq \epsilon_0$, the perturbed quantizer has the same or larger distortion.

Proof of Fact: Local opt \Rightarrow (*) and (**): If a quantizer does not satisfy (*) and (**), then it can be improved by small perturbations, so it is not a local optimum. Equivalently, the contrapositive says if it is a local optimum then it satisfies (*) and (**).

(*) and (**) \Rightarrow local opt: If a quantizer satisfies (*) and (**), then it is locally optimum because any small perturbation will cause it not to satisfy the (*) and (**) and, consequently, will be a worse quantizer. (this is kind of a fuzzy argument)

Iterative Design Algorithms

Lloyd-Design Algorithm (1957, 1982)

make an initial choice of quantization levels
use (*) to choose best thresholds for these levels
use (**) to choose best levels for these thresholds
use (*) to choose best thresholds for these levels
etc.
continue until (*) or (**) makes very little change.

Notes:

Each step decreases or maintains distortion, so distortion of algorithm converges.

Works well in practice

Levels usually converge. Convergence to a local minimum has been proven under certain conditions.

There is also a Lloyd-Max Algorithm:

J. Max, "Quantizing for minimum distortion," *IT*, 7-12, Mar. 1960.
Lloyd, 1957, 1982.

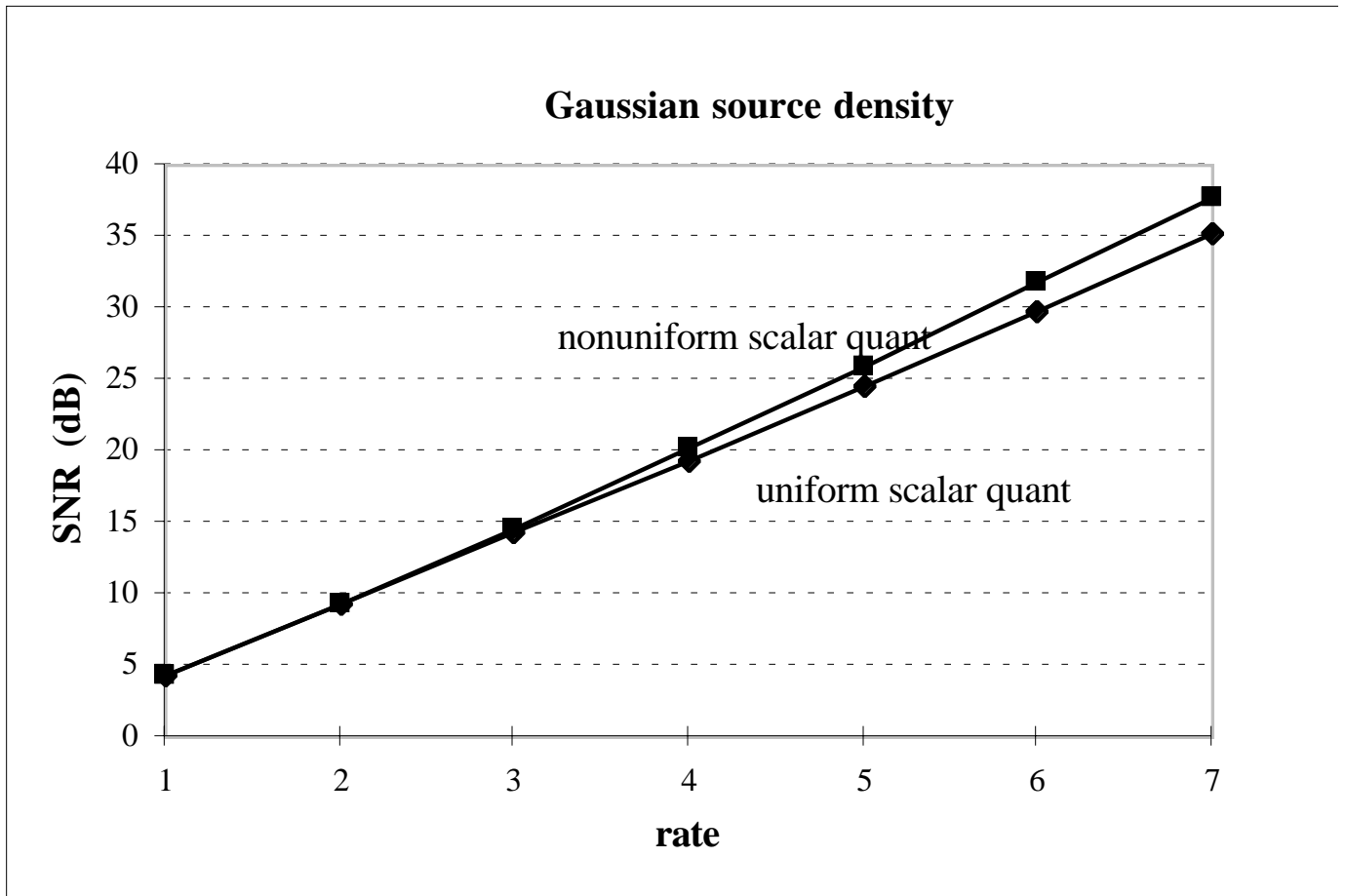
Optimum quantizers are often called Max or Lloyd-Max quantizers.

Optimal Scalar Quantizers

The figures on the following pages plot the SNR attained by scalar quantizers with minimum MSE for a given rate, where rate is defined by $R = \log_2 M$, for several different source densities.

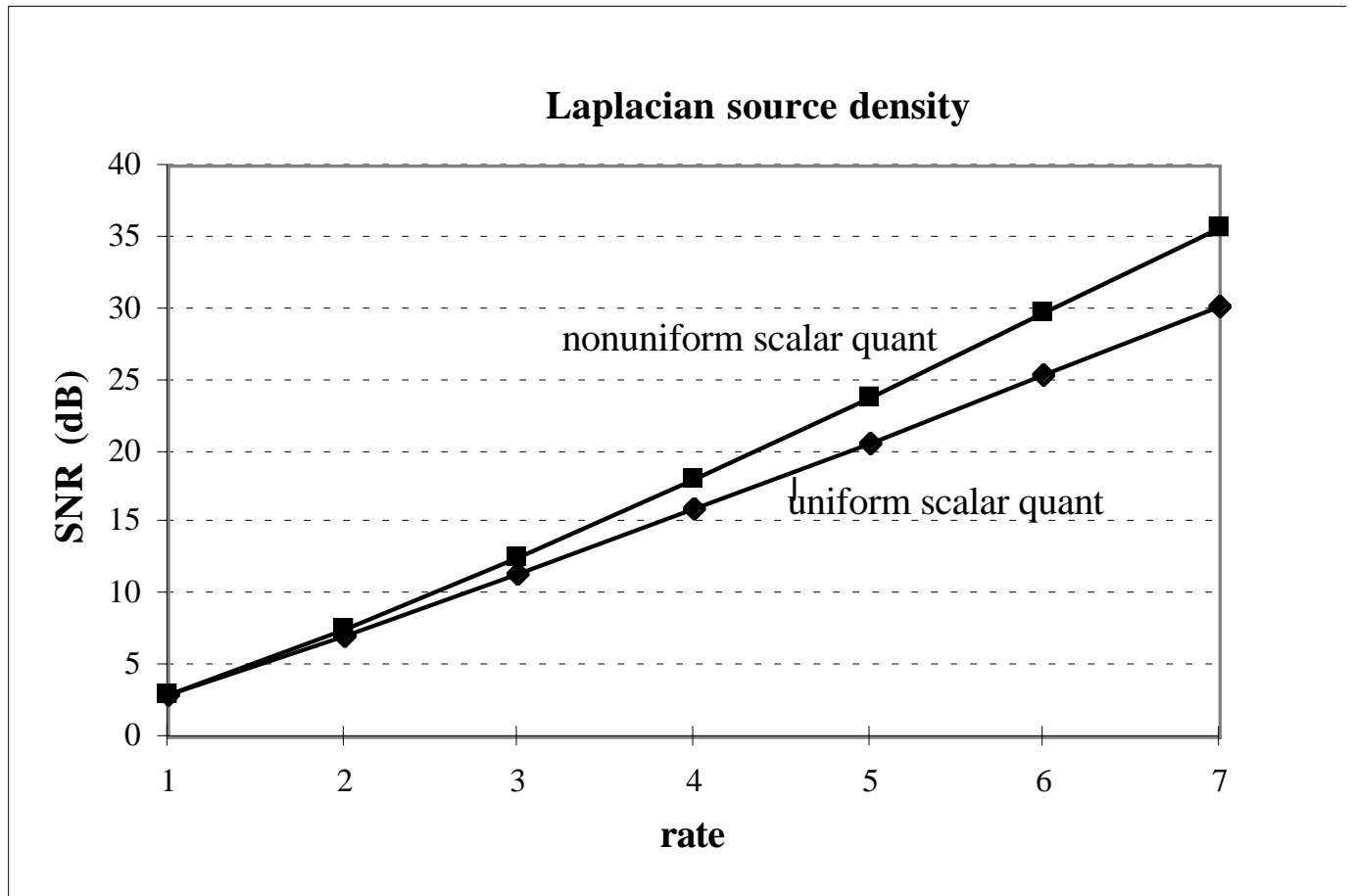
Also given are tables with the SNR's plotted, as well as those for optimized uniform scalar quantization.

Also given are plots comparing the SNR's for different densities.



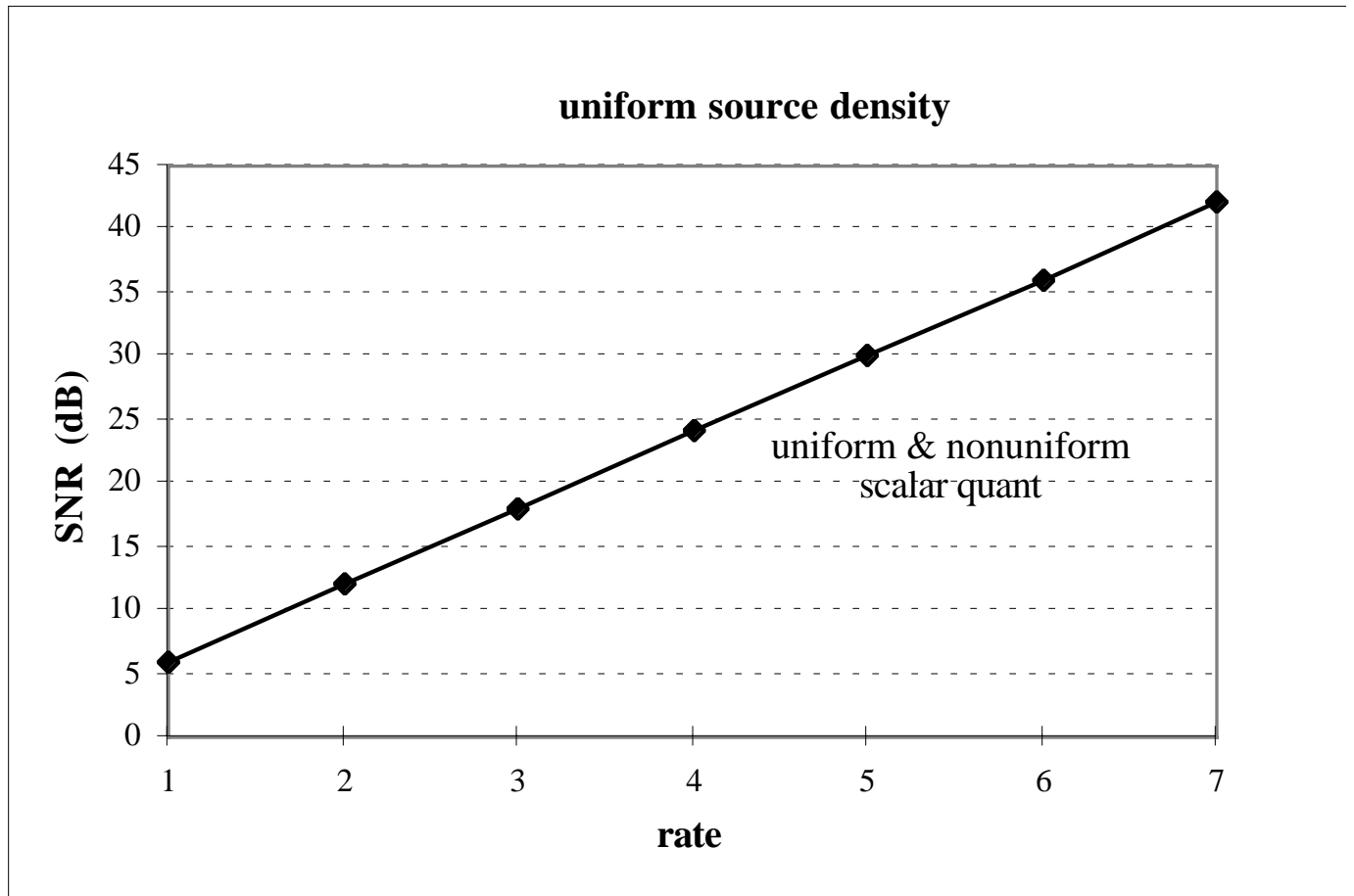
Gaussian density

Rate	Unif Q	Opt Q	Gain
1	4.4	4.4	0
2	9.25	9.3	0.05
3	14.27	14.62	0.35
4	19.38	20.22	0.84
5	24.57	26.01	1.44
6	29.83	31.89	2.06
7	35.13	37.81	2.68
8	40.34		



Laplacian

Rate	Unif Q	Opt Q	Gain	Panter Dite	diff
1	3.01	3.01	0	-0.51	3.52
2	7.07	7.54	0.47	5.51	2.03
3	11.44	12.64	1.2	11.53	1.11
4	15.96	18.13	2.17	17.55	0.58
5	20.6	23.87	3.27	23.57	0.30
6	25.36	29.74	4.38	29.59	0.15
7	30.23	35.69	5.46	35.61	0.08
8	35.14				



Uniform density

Rate	Unif Q	Opt Q	Gain
1	6.02	6.02	0
2	12.04	12.04	0
3	18.06	18.06	0
4	24.08	24.08	0
5	30.1	30.1	0
6	36.12	36.12	0
7	42.14	42.14	0
8	48.17	48.17	0

