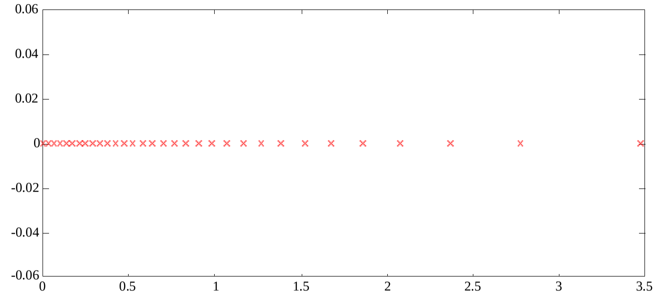
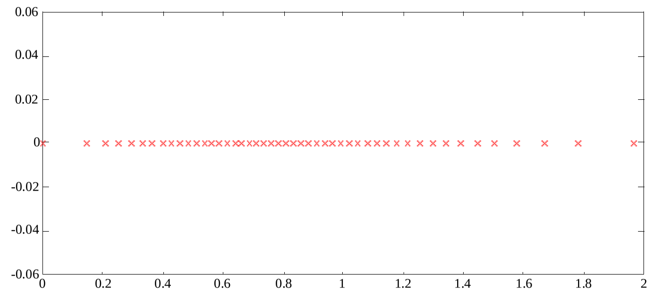
HIGH RESOLUTION ANALYSIS OF MSE OF NONUNIFORM QUANTIZERS (AKA ASYMPTOTIC ANALYSIS)

This will also lead to another design algorithm.

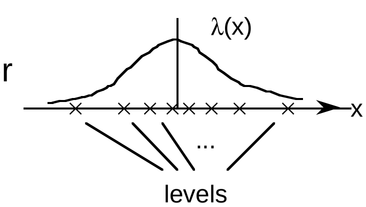
Consider how to roughly characterize quantizers with many levels and mostly small cells.





These quantizers differ in their densities of quantization levels/points.

The <u>point density</u> (aka level density or cell density) of a quantizer with many levels is a function $\lambda(x)$ such that for any a,b, a < b



(1) $\int_a^b \lambda(x)\,dx \cong \frac{\text{no. levels between a \& b}}{M}$

= fraction of levels betw. a & b

Ordinarily we assume $\lambda(x)$ is a fairly smooth function that at every x is either right or left continuous.

Other properties that follow from the above:

(2) $\int_a^b \lambda(x)\,dx \cong \frac{\text{no. cells between a \& b}}{M}$ = fraction cells betw. a & b

(3) $\lambda(x) \geq 0$, all x

(4) $\int_{-\infty}^{\infty} \lambda(x)\,dx = 1$

(5) $\lambda(x) \cong \frac{1}{M(t_i - t_{i-1})}$, when $t_{i-1} < x < t_i$, for most x, i.e. w. high prob.

(6) If b-a small, but large relative to cells

$\lambda(x)\,(b-a) \cong \frac{\text{no. levels between a \& b}}{M}$ = frac. levels betw. a & b

Notes:

- Usually, we don't employ a point density to describe a quantizer unless most cells are small, where "small" means that the probability density changes little across the cell and "most" means that the probability of the cells that are small is large. For example, the outermost cells are definitely not small, so they must have small probability.
- Usually, $\lambda(x)$ is a fairly smooth function that doesn't try to convey the detailed locations of levels and thresholds.
- Point density is generally an idealization or model. Often we pick a target point density and try to make our quantizer approximate it.
- We don't take (5) (with exact equality) as the definition because if we did, a quantizer would almost never have a specified point density.
- Note that (5) nearly implies the other properites. But not quite.
- When, as usual, $\lambda(x)$ is smooth, then (5) implies neighboring cells of the quantizer mostly have similar sizes; e.g. it rules out quantizers with cell sizes that alternate between large and small cells.

Proof (5) implies (1) and (2): Consider several cases

(a) Suppose $[a,b] = [t_{i-1},t_i]$ = one cell, then

$$\int_a^b \lambda(x)\,dx \cong \int_{t_{i-1}}^{t_i} \frac{1}{M(t_i-t_{i-1})}\,dx = (t_i-t_{i-1})\frac{1}{M(t_i-t_{i-1})} = \frac{1}{M}$$

(b) Suppose $[a,b] = [t_{m+1},t_{m+n}]$ = n cells , then

$$\int_a^b \lambda(x)\,dx \cong \int_{t_m}^{t_{m+n}} \lambda(x)\,dx \cong \sum_{i=m+1}^{m+n} \int_{t_{i-1}}^{t_i} \frac{1}{M(t_i-t_{i-1})}\,dx$$

$$= \sum_{i=m+1}^{m+n} (t_i-t_{i-1})\frac{1}{M(t_i-t_{i-1})} = \frac{n}{M}$$

(c) Suppose $t_{m-1} < a < t_m$ and $t_{m+n-1} < b < t_{m+n}$, so $[a,b]$ = n cells plus a little, then

$$\int_a^b \lambda(x)\,dx \cong \int_a^{t_m} \lambda(x)\,dx + \int_{t_m}^{t_{m+n}} \lambda(x)\,dx + \int_{t_{m+n}}^{b} \lambda(x)\,dx$$

$$\cong (t_m-a)\frac{1}{M(t_m-t_{m-1})} + \frac{n}{M} + (b-t_{m+n})\frac{1}{M(t_{m+n+1}-t_{m+n})}$$

$$\cong \frac{n}{M}$$

$\cong$ fraction of cells or levels between a and b,

because if M large, we can ignore the two "end terms"

Very important result:

Bennett's Integral Theorem

(W.R. Bennett, "Spectra of quantized signals," *BSTJ,* 27, 446-472, July, 1948.)

If a quantizer has mostly small cells, point density approxmately $\lambda(x)$, neighboring cells have similar sizes, and levels are at the centers of the cells (approximately), then

$$D \cong \frac{1}{12M^2} \int_{-\infty}^{\infty} \frac{f_X(x)}{\lambda^2(x)}\, dx$$

Notes:

- The meaning of "mostly small cells" was described earlier.
- This formula shows that for quantizers with a given point density λ, distortion decreases as $1/M^2$, or equivalently, SNR increases as 6 dB/bit.
- The Bennett intengral formula is usually fairly accurate for good (including optimal) quantizers when $M \geq 8$.
- This formula identifies point density as a key characteristic.
- Bennett's original derivation of this was in terms of companding functions, not point densities.

DERIVATION OF BENNETT'S INTEGRAL

$$D \cong \int_{\text{gran region}} (x-Q(x))^2 f_X(x)\, dx \;=\; \text{granular distortion}$$

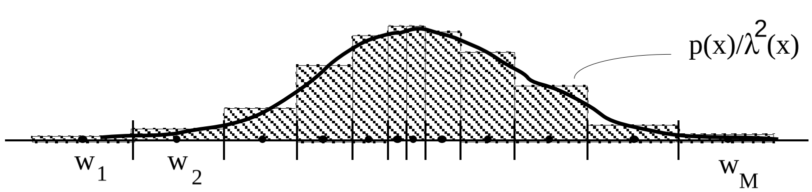
(we ignore overload dist'n because most cells are small)

$$= \sum_{i=1}^{M} \int_{t_{i-1}}^{t_i} (x-w_i)^2 f_X(x)\, dx \;\cong\; \sum_{i=1}^{M} f_X(w_i) \int_{t_{i-1}}^{t_i} (x-w_i)^2\, dx$$

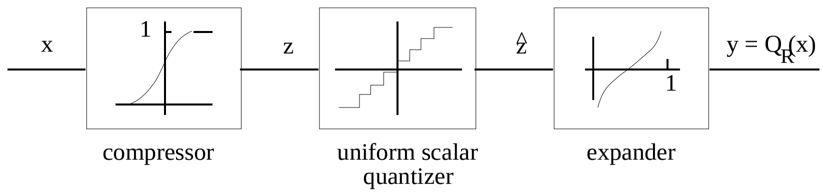
$$= \sum_{i=1}^{M} f_X(w_i) \frac{(\mathrm{len}(S_i))^3}{12} \;=\; \frac{1}{12M^2} \sum_{i=1}^{M} f_X(w_i)\, (M\,\mathrm{len}(S_i))^2\, \mathrm{len}(S_i)$$

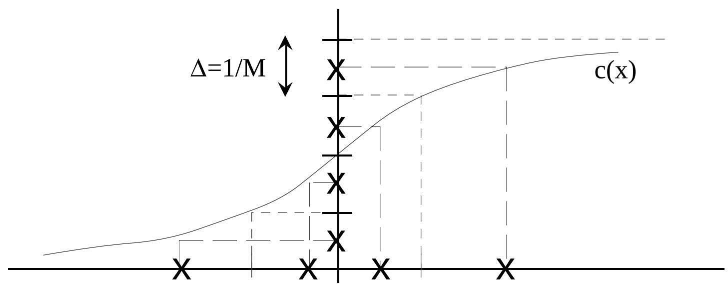
$$\cong \frac{1}{12M^2} \sum_{i=1}^{M} f_X(w_i) \frac{1}{\lambda^2(w_i)}\, \mathrm{len}(S_i)$$

$$\cong \frac{1}{12M^2} \int_{-\infty}^{\infty} f_X(x) \frac{1}{\lambda^2(x)}\, dx$$



THE POINT DENSITY OF A COMPANDER





- $t_i = c^{-1}(\frac{i}{M})$, $i = 1,\ldots,M-1$; $y_i = c^{-1}(\frac{i}{M} - \frac{1}{2M})$, $i = 1,\ldots,M$
- If x is contained in the cell (t_{i-1},t_i) then the width of this cell is $(t_i - t_{i-1}) \cong \frac{1/M}{c'(x)}$
- $\lambda(x) \cong \frac{1}{M(t_i - t_{i-1})} \cong c'(x)$

In words, the point density of a compander is the derivative of the companding function.

DESIGN TO ACHIEVE A CERTAIN POINT DENSITY

Suppose you wish to design a quantizer with a given point density $\lambda(x)$. Then choose companding function

$$c(x) = \int_{-\infty}^{x} \lambda(x')\, dx'$$

and let the thresholds and levels be

$$t_i = c^{-1}(\frac{i}{M}),\ \ i = 1,\ldots,M-1$$

$$y_i = c^{-1}(\frac{i}{M} - \frac{1}{2M}),\ \ i = 1,\ldots,M$$

COMPARISON OF MSE PREDICTED BY BENNETT WITH ACTUAL MSE.

In the following tables, the quantizer's were designed via the companding method described above to have the point density that will later be shown to be optimal. All SNR's are in dB.

Gaussian

M	SNR from Bennett	actual SNR	diff.
2	1.68	3.01	1.33
4	7.70	8.37	.67
8	13.72	14.07	.35
16	19.74	19.92	.18
32	25.76	25.85	.09
64	31.78	31.83	.05
128	37.80	37.82	.02

Laplacian

M	SNR from Bennett	actual SNR	diff.
2	-0.51	-3.45	-2.94
4	5.51	6.16	.65
8	11.53	11.93	.4
16	17.55	17.76	.21
32	23.57	23.68	.11
64	29.59	29.64	.05
128	35.61	35.64	.03

More Notes:

- A rigorous derivation of Bennett's integral considers a sequence of quantizers $Q_1, Q_2, \ldots$ where
  (a) Q_M has levels $y_1^{(M)}, \ldots, y_M^{(M)}$ and thresholds $t_1^{(M)}, \ldots, t_{M-1}^{(M)}$
  (b) The quantizers have point density approaching $\lambda(x)$ in the sense that
  $$\frac{1}{M \times (\text{length of cell of } Q_M \text{ containing } X)} \to \lambda(X) \text{ in prob.}$$
  (c) The levels asymptotically approach the cell centers in the sense that
  $$\frac{|Q_M(X) - \text{center pointt of cell containing } x|}{\text{length of cell of } Q_M \text{ containing } x} \to 0 \text{ in prob.}$$
- A theorem shows that if $E[X^{2+\delta}] < \infty$ for some $\delta > 0$, then
  $$\lim_{M\to\infty} M^2 D(Q_M) = \frac{1}{12} \int \frac{1}{\lambda^2(x)} p(x)\, dx$$
- Reference: J.A. Bucklew & G.L. Wise, "Multidimensional asymptotic quantization theory with the rth power distortion measures," IEEE Trans. Inform. Thy., vol. IT-28, pp. 239-247, Mar. 1982.

- If each level is located at the right edge of its cell, then $\frac{1}{12}$ in Bennett's integral is replaced by $\frac{1}{3}$. (This is a good excercise.)
- If adjacent cell sizes are too different (e.g. if they alternate large and small), then although one could define $\lambda(x) = (M \times \text{avg cell size near } x)^{-1}$, the above formula won't hold.
- Example: Suppose a quantizer has M cells whose sizes alternate between $\Delta = 2/(3M)$ and $2\Delta = 4/(3M)$ and the source has a uniform pdf on $[0,1]$. Then

$$D = \frac{1}{3}\frac{\Delta^2}{12} + \frac{2}{3}\frac{(2\Delta)^2}{12} = \frac{\Delta^2}{4} = \frac{1}{4}\frac{4}{9M^2} = \frac{1}{9M^2}$$

Whereas taking the macro viewpoint that $\lambda(x) = 1$ and plugging into Bennett's integral, gives

$$D = \frac{1}{12M^2}$$

Correct use of Bennett's integral. Note that the distortion is not changed if all the small cells were moved to the left and the large cells were moved to the right, in which case

$$\lambda(x) = \begin{cases} \frac{3}{2}, & x \le \frac{1}{3} \\ \frac{3}{4}, & x \ge \frac{1}{3} \end{cases} \quad \text{and} \quad D = \frac{1}{12M^2}\left(\frac{1}{3}\frac{4}{9} + \frac{2}{3}\frac{16}{9}\right) = \frac{1}{9M^2}$$

- **Bennett's integral is convex cup in $\boldsymbol{\lambda}$**; i.e.

$$B\left(\tfrac{1}{2}\lambda_1 + \tfrac{1}{2}\lambda_2\right) \le \frac{1}{2}B(\lambda_1) + \frac{1}{2}B(\lambda_2)$$

where $B(\lambda) = \frac{1}{12M^2}\int_{-\infty}^{\infty}\frac{f_X(x)}{\lambda^2(x)}\,dx$

This is because $\frac{1}{\lambda^2}$ is convex cup, and so

$$\frac{1}{\left(\frac{1}{2}\lambda_1(x) + \frac{1}{2}\lambda_2(x)\right)^2} \le \frac{1}{2}\frac{1}{\lambda_1^2(x)} + \frac{1}{2}\frac{1}{\lambda_2^2(x)}$$

THE OPTIMAL POINT DENSITY

Questions:

- What point density minimizes Bennett's integral?

  (An optimal quantizer with large M will have this as its point density, approximately.)

- What is the distortion of optimal quantizers?

- For large M the best quantizers for the source density $f_X(x)$ have point density

$$\lambda^*(x) = \frac{f_X^{1/3}(x)}{\int_{-\infty}^{\infty} f_X^{1/3}(x)\,dx}$$

- For large M the best quantizers for the source density $f_X(x)$ have MSE given, approximately, by the <u>Panter-Dite Formula</u>

$$\delta_{sq}(M) \cong \frac{\sigma^2 \beta}{12M^2}$$

where σ^2 is the source variance and

$$\beta \triangleq \frac{1}{\sigma^2}\left(\int_{-\infty}^{\infty} f_X^{1/3}(x)\,dx\right)^3 = \text{Panter-Dite factor}$$

is the term that depends on the source density.

- One can straightforwardly show that β is not affected by a scaling of X; i.e. it depends only on the shape of the source density.
- Reference: P.F. Panter and W. Dite, "Quantization distortion in pulse count modulation with nonuniform spacing of levels," *Proc. IRE,* 44-48, Jan. 1951.
- Equivalently, for large R, the best quantizers with rate R have MSE given by the OPTA function

$$\delta_{sq}(R) \cong \frac{\sigma^2\beta}{12} 2^{-2R} \quad \text{(another version of Panter-Dite formula)}$$

- Equivalently, the best quantizers with rate R (large) have signal-to-noise ratio

$$S_{sq}(R) = 10 \log_{10} \frac{\sigma^2}{\delta_{sq}(R)} \cong 6.02\,R + 10 \log_{10} \frac{12}{\beta}$$

This shows that with the best quantizers, SNR increases 6 dB for every one bit increase in the rate R. This is called the "6 dB per bit" rule.

- Values of the Panter-Dite Factor β for common densities

	uniform	Gaussian	Laplacian	Gamma
β	12	$6\sqrt{3}\pi = 32.65$	54	$48\sqrt{3/\pi}\,\Gamma^3(\frac{5}{6}) = 67.463$
$10\log_{10}\frac{12}{\beta}$	0 dB	-4.347 db	-6.532 dB	-7.499 dB

where $\Gamma(\,)$ is the gamma function.

DERIVATION OF THE PANTER-DITE FORMULA

We find the function λ such that $\lambda(x) \geq 0$, $\int_{-\infty}^{\infty} \lambda(x)\,dx = 1$, and $\int_{-\infty}^{\infty} \frac{f_X(x)}{\lambda^2(x)}\,dx$ is as small as possible. We minimize the integral using Holder's inequality (could also use calculus of variations).

Holder's inequality:

Given functions f, g & positive numbers p, q such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int f(x)\, g(x)\, dx \leq \left(\int |f(x)|^p dx\right)^{1/p} \left(\int |g(x)|^q dx\right)^{1/q}$$

with equality iff there is a constant c such that $|f|^p = c|g|^q$.

Approach: Choose f, g, p, q so $\int |f(x)|^p dx = \int_{-\infty}^{\infty} \frac{f_X(x)}{\lambda^2(x)})\, dx$. Then

$$\int_{-\infty}^{\infty} \frac{f_X(x)}{\lambda^2(x)}\,dx = \int |f(x)|^p dx \geq \left(\int f(x)\, g(x)\, dx\right)^p \Big/ \left(\int |g(x)|^q dx\right)^{p/q}$$

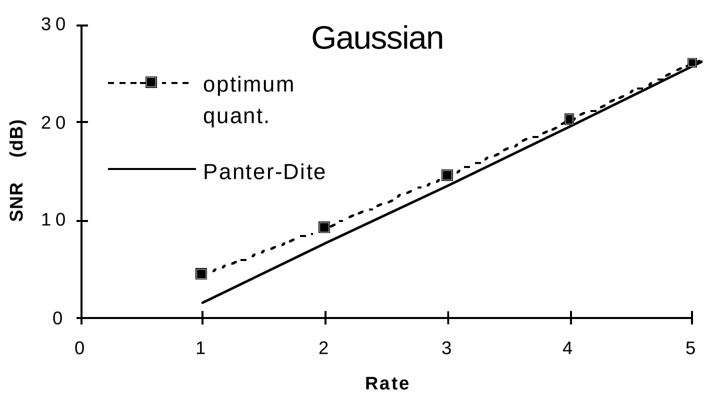
with equality iff there is constant c such that $|f(x)|^p = c|g(x)|^q$ for all x.

Let: $p = 3$, $q = 3/2$, $f(x) = f_X^{1/3}(x)\,\lambda^{-2/3}(x)$, $g(x) = \lambda^{2/3}(x)$

Then $\int_{-\infty}^{\infty} \frac{f_X(x)}{\lambda^2(x)}\,dx = \int |f(x)|^p dx \geq \left(\int f_X^{1/3}(x)\,dx\right)^3 \Big/ \left(\int |\lambda(x)|\,dx\right)^2 = \left(\int f_X^{1/3}(x)\,dx\right)^3 = \sigma^2\beta$

with equality iff $|f_X^{1/3}(x)\lambda^{-2/3}(x)|^3 = c|\lambda^{2/3}(x)|^{3/2}$ iff $|f_X(x)| = c\,\lambda^3(x)$ iff $\lambda(x) = c^{-1/3} f^{1/3}(x)$

PREDICTED VS ACTUAL OPTA FUNCTION

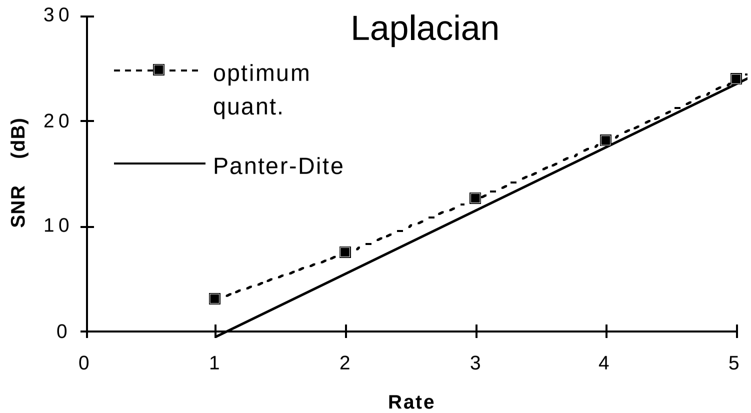


Gaussian (SNR in dB)

M	predicted opta	actual opta	difference
2	1.68	4.4	2.72
4	7.70	9.3	1.60
8	13.72	14.62	0.90
16	19.74	20.22	0.48
32	25.76	26.01	0.25
64	31.78	31.89	0.11
128	37.80	37.81	0.01

"Actual" SNR's from Jayant and Noll

PREDICTED VS ACTUAL OPTA FUNCTION



Laplacian (SNR in dB)

M	predicted opta	actual opta	difference
2	-0.51	3.01	3.52
4	5.51	7.54	2.03
8	11.53	12.64	1.11
16	17.55	18.13	0.58
32	23.57	23.87	0.30
64	29.59	29.74	0.15
128	35.61	35.69	0.08

"Actual" SNR's from Jayant and Noll

COMPARISON OF

- Panter-Dite prediction of the SNR OPTA
- SNR of quantizers designed by companding method to have optimal pt. density.
- Actual SNR OPTA

Gaussian

M	predicted OPTA from P-D formula	quantizer with opt pt. density	actual opta
2	1.68	3.01	4.4
4	7.70	8.37	9.3
8	13.72	14.07	14.62
16	19.74	19.92	20.22
32	25.76	25.85	26.01
64	31.78	31.83	31.89
128	37.80	37.82	37.81

Laplacian

N	predicted OPTA from P-D formula	quantizer with opt. pt. density	actual opta
2	-0.51	-3.45	3.01
4	5.51	6.16	7.54
8	11.53	11.93	12.64
16	17.55	17.76	18.13
32	23.57	23.68	23.87
64	29.59	29.64	29.74
128	35.61	35.64	35.69

Interesting Asymptotic Properties of Optimal Quantizers

For optimal quantizers with many levels

- Cells are smaller where the density $f_X(x)$ is larger.

  Cell containing x has width: $|t_i - t_{i-1}| \cong \frac{1}{M\lambda^*(x)} = \frac{1}{M c f_X^{1/3}(x)}$

- Cells have more probability where $f_X(x)$ is larger, even though cells are smaller

  Probability of cell containing x:

$$P_i = \Pr(t_{i-1} < X < t_i) \cong |t_i - t_{i-1}| f_X(x) \cong \frac{1}{M c f_X(x)^{1/3}} f_X(x) = \frac{1}{Mc} f_X^{2/3}(x)$$

- Each cell contributes roughly the same amount to the distortion.

  The contribution to distortion of the cell containing x is

$$\int_{t_{i-1}}^{t_i} (x'-w_i)^2 f_X(x')\, dx' \cong f_X(x) \frac{(t_i - t_{i-1})^3}{12} \cong f_X(x) \frac{1}{12M^3c^3 f_X(x)} = \frac{1}{12M^3c^3}$$