Addendum to DPCM and Transform Coding Lectures:

This addendum demonstrates the relationship between the determinants of covariance matrices of a widesense stationary random process and the n-step prediction MSE, which we have used in our discussions of transform coding, DPCM and vector quantization.

Fact: If X is a wide-sense stationary, zero-mean random process, then

$$\left|K^{(n+1)}\right| = M_n \left|K^{(n)}\right|$$

where M_n denotes the mean-squared error of the best linear predictor for X_i from $X_{i-1},\ldots,X_{i-n}$, $K^{(n)}$ and $K^{(n+1)}$ denote, respectively, the covariance matrices of $X_1,\ldots,X_n$ and $X_1,\ldots,X_{n+1}$ with determinants $\left|K^{(n)}\right|$ and $\left|K^{(n+1)}\right|$, respectively.

Proof: We will use some standard facts regarding determinants and Cramer's rule for solving systems of linear equations.

A. We begin by finding an expression for $\left|K^{(n+1)}\right|$.

Let $r_i = E[X_n X_{n+i}]$ and $\underline{r} = [r_1, r_2, \ldots, r_n]^t$. (Note that $r_0 = \mathrm{var}(X)$.) In terms of the r_i's,

$$K^{(n+1)} = \begin{bmatrix} r_0 & r_1 & . \; . \; . & r_{n-1} & r_n \\ r_1 & r_0 & . \; . \; . & r_{n-2} & r_{n-1} \\ & & . & & \\ r_{n-1} & r_{n-2} & . \; . \; . & r_0 & r_1 \\ r_n & r_{n-1} & . \; . \; . & r_1 & r_0 \end{bmatrix} \quad \text{and} \quad K^{(n)} = \begin{bmatrix} r_0 & r_1 & . \; . \; . & r_{n-1} \\ r_1 & r_0 & . \; . \; . & r_{n-1} \\ & & . & \\ r_{n-1} & r_{n-2} & . \; . \; . & r_0 \end{bmatrix}.$$

We now expand the determinant of $K^{(n+1)}$ along the first row of the matrix:

$$\begin{aligned} \left|K^{(n+1)}\right| &= r_0 \left|L_1\right| - r_1 \left|L_2\right| + r_2 \left|L_3\right| \ldots + (-1)^n r_n \left|L_{n+1}\right| \\ &= r_0 \left|K^{(n)}\right| + \sum_{i=1}^{n} r_i \left|L_{i+1}\right| (-1)^i \qquad (1) \end{aligned}$$

where L_i is $K^{(n+1)}$ with the first row and ith column removed (L_i is called a *minor* of $K^{(n+1)}$) and where it is easy to see that $\left|L_1\right| = \left|K^{(n)}\right|$. ($L_i$ is called a minor of $K^{(n+1)}$.)

B. We now find an expression for $M_n \left|K^{(n)}\right|$ that can be compared to the expression (1) just found for $\left|K^{(n+1)}\right|$.

Recall that in the DPCM notes it was shown that

$$M_n = r_0 - \underline{r}^t \left(K^{(n)}\right)^{-1} \underline{r} \quad . \qquad (2)$$

Letting

$$\underline{s} = [s_1,s_2,\ldots,s_n]^t = (K^{(n)})^{-1}\underline{r}$$

the expression for M_n becomes

$$M_n = r_0 - \underline{r}^t\,\underline{s}$$

Cramer's rule for solving the equation $\underline{s}\,K^{(n)} = \underline{r}$, for $\underline{s} = [s_1,s_2,\ldots,s_n]^t$ gives

$$s_i = \frac{|B_i|}{|K^{(n)}|}\ , \ i = 1,...,n$$

where B_i is $K^{(n)}$ with its ith column replaced by $\underline{r}$. Therefore the second term one the right side of (2) becomes

$$\underline{r}^t\,(K^{(n)})^{-1}\underline{r} = \underline{r}^t\,\underline{s} = \frac{1}{|K^{(n)}|}\sum_{i=1}^{n} r_i\,|B_i|.$$

and plugging the above into (2) we find

$$M_n\,|K^{(n)}| = r_0\,|K^{(n)}| - \sum_{i=1}^{n} r_i\,|B_i|\ . \tag{3}$$

C. We now show that expressions (1) and (3) for $|K^{(n+1)}|$ and $M_n\,|K^{(n)}|$ are equal to one another. To do so, it suffices show that

$$-|B_i| = |L_{i+1}|(-1)^i,\ i = 1,\ldots,n\ .$$

To do so, observe that by its definition, the first column of L_{i+1} is $\underline{r}$ and the remaining columns are the columns of $K^{(n)}$ except that the ith column of $K^{(n)}$ is omitted, i.e.

$$L_{i+1} = \left[\underline{r}\ ,\ \tilde{K}_i^{(n)}\right],\ i =1,\ldots,n,$$

where $\tilde{K}_i^{(n)}$ is $K^{(n)}$ with its ith column removed. We now observe that moving the first column of L_{i+1} to the right (i-1) places creates the matrix B_i. Since moving a column to the right one place multiplies the determinant by -1, we have

$$|B_i| = |L_{i+1}|(-1)^{i-1} = -|L_{i+1}|(-1)^i,\ i=1,\ldots,n,$$

which is just what we needed to show, and which completes the proof.