1. A binary IID source has $p(0) = .995$ and $p(1) = .005$.
   (a) Find a fixed-to-variable length block lossless source code with rate no larger than 0.4. (It should be as simple as possible.)
   (b) Is it possible to find a fixed-to-variable length block lossless source code with rate less than 0.1? If so, what can you say about how large its input block length must be?

2. Suppose we are told that a certain stationary source $\{X_k\}$ with alphabet $A = \{1,2,\ldots,M\}$ has $R^*_3 = .95$ and $R^*_4 = .92$.
   Where $R^*_k$ denotes the least rate of any block to variable length prefix code with input blocklength $k$. What can you deduce about the values of $H_\infty$ and $H_k$ for $k = 1,2,3,\ldots$? In other words, using the given information, find the tightest possible bounds to $H_\infty$ and to each $H_k$.

3. Consider the "runlength" code shown below. This is a variable-length to fixed-length code, unlike the fixed-length to variable-length we have considered in class. Let the source be the IID source with $p(0) = .995$ and $p(1) = .005$.

<table>
<thead>
<tr>
<th>source sequence</th>
<th>run length</th>
<th>binary codeword</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>000</td>
</tr>
<tr>
<td>01</td>
<td>1</td>
<td>001</td>
</tr>
<tr>
<td>001</td>
<td>2</td>
<td>010</td>
</tr>
<tr>
<td>0001</td>
<td>3</td>
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<td>00001</td>
<td>4</td>
<td>100</td>
</tr>
<tr>
<td>000001</td>
<td>5</td>
<td>101</td>
</tr>
<tr>
<td>0000001</td>
<td>6</td>
<td>110</td>
</tr>
<tr>
<td>0000000</td>
<td>7</td>
<td>111</td>
</tr>
</tbody>
</table>
   (a) Explain why this code is uniquely encodable and decodable.
   (b) Find the average length of the encoded source sequences.
   (c) Find the rate of this code.
   (d) Compare the rate and complexity of this code to that found in Problem 1a.

4. Let $\{Z_k\}$ be a binary IID source with alphabet $A_Z = \{0,1\}$ and $p_Z(1) = q$. Suppose $\{X_k\}$ is a stationary binary source with alphabet $A_X = \{0,1\}$, with $p_{X_k}(1) = 1/2$, $p_{X_k,X_{k-1}}(0,0) = p_{X_k,X_{k-1}}(0,1) = p_{X_k,X_{k-1}}(1,0) = p_{X_k,X_{k-1}}(1,1)$, and with
   $X_k = X_{k-1} \oplus X_{k-2} \oplus Z_k$,
   where $\oplus$ denotes modulo 2 addition, and where $Z_k$ is independent of $X_{k-1}, X_{k-2}, X_{k-3}, \ldots$.
   (a) Show that the $\{X_k\}$ is $N$th-order Markov for some appropriately chosen $N$.
   (b) Find an expression for $H_k$ and $H_\infty$.
   For the remaining parts assume $q = 0.1$.
   (c) Make a sketch showing how $H_k$ approaches $H_\infty$ as $k$ increases.
   (d) Find minimum rate FL-VL block prefix codes for block lengths 1, 2, and 3. Compute their rates and plot them vs. source length on the sketch of part (c).
   (e) Can you design a conditional code that is better than those in (d), taking into account rate and complexity?
5. Let $X$ be a random variable with pdf $f_X(x)$. Consider a scalar quantizer for $X$ with thresholds \{t_0, t_1, ..., t_M\}, levels \{y_1, ..., y_M\}, and MSE $D_X$. Let $Y = aX + b$, where $a$ and $b$ are constants. Consider quantizing $Y$ with the quantizer with thresholds \{at_0 + b, at_1 + b, ..., at_M + b\}, levels \{ay_1 + b, ..., ay_M + b\} and MSE $D_Y$.

(a) Show that
$$D_Y = a^2 D_X$$

(b) Show that if the quantizer for $X$ has smallest MSE of any quantizer for $X$ with $M$ levels, then the quantizer for $Y$ has smallest MSE of any quantizer for $Y$ with $M$ levels, and vice versa.

(c) Show that the smallest possible MSE of any quantizer for $Y$ with $M$ levels is $a^2$ times the smallest possible MSE of any quantizer for $X$ with $M$ levels. (One may conclude that for two pdf's with the same shape, the minimal MSE is proportional to variance. This is a very important conclusion.)