Summary of Asymptotic Analysis of Optimal Uniform Scalar Quantization[1]

For Gaussian Density

For a Gaussian density source density with variance σ^2, a large value of M, and the symmetric uniform scalar quantizer with M levels and smallest MSE, the formulas below give useful approximations to (1) the support region $[-L_M, L_M]$, (2) the step size $\Delta_M = 2L_M/M$, (3) the MSE D_M, and (4) the signal-to-noise ratio S(R), assuming rate $R = \log_2 M$:

$$L_M \cong 2\sigma\sqrt{\ln M}$$

$$\Delta_M = \frac{2L_M}{M} \cong \sigma\frac{4\sqrt{\ln M}}{M}$$

$$D_M \cong \frac{\Delta_M^2}{12} \cong \frac{4}{3}\sigma^2\frac{\ln M}{M^2}$$

$$S(R) = 10\log_{10}\frac{\sigma^2}{D_M} \cong 10\log_{10}\frac{3}{4\ln 2} + 6.02\,R - 10\log_{10}R$$

$$= 6.02\,R - 10\log_{10}R + 0.34 \text{ dB}$$

A more accurate approximation is

$$L_M \cong \sigma\left(4\ln M - 3\ln\ln M - \ln\frac{32\pi}{9}\right)^{1/2}$$

Tables and figures given later compare the values given by these expressions with the actual values, so one can get a feeling for their accuracy..

For Generalized Gamma Densities

As described on the following pages, these formulas for the Gaussian density have been extended to generalized Gamma densities.

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[1]Reference: D. Hui and D.L. Neuhoff, "Asymptotic Analysis of Optimal Fixed-Rate Uniform Scalar Quantization," *IEEE Trans. Information Theory*, vol. 47, pp. 957-977, Mar. 2001, http://ieeexplore.ieee.org/iel5/18/19784/00915652.pdf .

Generalized Gamma Densities

A generalized Gamma density with variance σ^2 has the form

$$p(x) = \mu|x|^{\beta} \exp\left\{-\lambda|x|^{\alpha}\right\}, \quad -\infty < x < \infty,$$

where $\alpha > 0$ is the exponential decay parameter, $\beta > -1$ is the power of $|x|$, and μ, λ are constants chosen so the density integrates to one and has variance σ^2. In particular,

$$\mu = \frac{1}{\sigma^{\beta+1}} \frac{\Gamma\left((\beta+3)/\alpha\right)^{(\beta+1)/2}}{\Gamma\left((\beta+1)/\alpha\right)^{(\beta+3)/2}}$$

$$\lambda = \left[\frac{1}{\sigma}\sqrt{\frac{\Gamma\left((\beta+3)/\alpha\right)}{\Gamma\left((\beta+1)/\alpha\right)}}\right]^{\alpha}.$$

The following table gives the parameters of some common special cases of generalized Gamma.

Source	α	β	λ	μ
Gaussian	2	0	$\frac{1}{2\sigma^2}$	$\frac{1}{\sqrt{2\pi}\sigma}$
Laplacian	1	0	$\frac{\sqrt{2}}{\sigma}$	$\frac{1}{\sqrt{2}\sigma}$
2-sided Rayleigh	2	1	$\frac{1}{\sigma^2}$	$\frac{1}{\sigma^2}$
Gamma	1	-1/2	$\frac{\sqrt{3}}{2\sigma}$	$\frac{3^{1/4}}{\sqrt{8\pi\sigma}}$

Note:

If $\beta=0$ and α tends to infinity, then the GΓ density approaches the uniform density

Theorem: For a generalized Gamma density with variance σ^2, the best symmetric scalar quantizer with M levels has support region $[-L_M, L_M]$, step size $\Delta_M = 2L_M/M$ and distortion D_M satisfying

(1) $$\lim_{M\to\infty} \frac{L_M}{(\ln M)^{1/\alpha}} = \left(\frac{2}{\lambda}\right)^{1/\alpha}; \qquad \text{i.e. } L_M \cong \left(\frac{2}{\lambda}\ln M\right)^{1/\alpha} \text{ when } M \text{ is large}$$

(2) $$\lim_{M\to\infty} \frac{\Delta_M}{(\ln M)^{1/\alpha}/M} = 2\left(\frac{2}{\lambda}\right)^{1/\alpha}; \qquad \text{i.e. } \Delta_M \cong \frac{2}{M}\left(\frac{2}{\lambda}\ln M\right)^{1/\alpha} \text{ when } M \text{ is large}$$

(3) $$\lim_{M\to\infty}\left\{L_M^{\alpha} - \left[\frac{2\ln M}{\lambda} - \left(2-\frac{1+\beta}{\alpha}\right)\frac{\ln\ln M}{\lambda} - \frac{1}{\lambda}\ln\left(\frac{2^{1-(1+\beta)/\alpha}\alpha^2\lambda^{(1+\beta)/\alpha}}{3\mu}\right)\right]\right\} = 0$$

$$\text{i.e. } L_M \cong \left[\frac{2\ln M}{\lambda} - \left(2-\frac{1+\beta}{\alpha}\right)\frac{\ln\ln M}{\lambda} - \frac{1}{\lambda}\ln\left(\frac{2^{1-(1+\beta)/\alpha}\alpha^2\lambda^{(1+\beta)/\alpha}}{3\mu}\right)\right]^{1/\alpha}$$

(4) $$\lim_{M\to\infty} \frac{D_M}{\Delta_M^2/12} = 1; \qquad \text{i.e. } D_M \cong D_{gran,M} \cong \frac{\Delta_M^2}{12}, \quad \text{also } \frac{D_{over,M}}{D_{gran,M}} \to 0$$

(5) $$\lim_{M\to\infty} \frac{M^2}{(\ln M)^{2/\alpha}} D_M = \frac{1}{3}\left(\frac{2}{\lambda}\right)^{2/\alpha}; \qquad \text{i.e. } D_M \cong \frac{1}{3}\left(\frac{2}{\lambda}\right)^{2/\alpha}\frac{(\ln M)^{2/\alpha}}{M^2}$$

$$\text{or} \qquad S(M) = 10\log_{10}\frac{\sigma^2}{D_M} \cong 10\log_{10} 3\left(\frac{\lambda}{2}\right)^{2/\alpha} + 20\log_{10} M - \frac{20}{\alpha}\log_{10}\ln M$$

$$\text{or} \qquad S(R) \cong 10\log_{10} 3\sigma^2\left(\frac{\lambda}{2\ln 2}\right)^{2/\alpha} + 6.02R - \frac{20}{\alpha}\log_{10} R$$

Note:

As α tends to infinity and $\beta = 0$, the GΓ density approaches the uniform density, and the constant appearing in the limiting expression in (1) tends to $\sqrt{3}\sigma$, which is the optimal support length for a uniform density with variance σ^2.

Comparison of Asymptotic Formulas with Actual Values

Comparisons of the actual values of L_M and $S(R)$ found by numerical optimization algorithms with the values predicted by the formulas above are given in Table 1 and Figures 1-8.

L_M denotes the true optimal value, which is found by an iterative algorithm. $\hat{L}_{M,1}$ and $\hat{L}_{M,2}$ are the estimates based on (1) and (3), respectively. $\hat{L}_{M,3}$ is another estimate described in the paper.

S^*, S_0 and S_1 denote estimates of SNR based on the formula $D \cong \Delta^2/12 = (4L^2/12M^2)$, i.e. it assumes granular distortion alone. The values of L are L_M, $\hat{L}_{M,1}$, and $\hat{L}_{M,3}$, respectively. S_2 is another estimate described in the paper that adds a term for overload distortion.

Source	Rate	M	Optimal Δ_M	Optimal L_M	$\hat{L}_{M,1}$	$\hat{L}_{M,2}$	$\hat{L}_{M,3}$	Actual SNR	S^*	S_0	S_1	S_2
G	4	16	0.3352	2.68	3.33	2.37	2.69	19.38	20.29	18.4	20.25	19.35
	6	64	0.1041	3.33	4.08	3.15	3.32	29.83	30.45	28.68	30.47	29.69
	8	256	0.0308	3.94	4.71	3.82	3.92	40.34	41.03	39.48	41.07	40.44
L	4	16	0.4610	3.69	3.92	3.49	3.77	15.96	17.52	16.99	17.32	16.32
	6	64	0.1657	5.30	5.88	5.16	5.33	25.36	26.41	25.51	26.36	25.46
	8	256	0.0549	7.03	7.84	6.92	7.03	35.14	36.00	35.05	36.00	35.24
R	4	16	0.2747	2.20	2.35	2.06	2.21	21.27	22.01	21.43	21.97	21.47
	6	64	0.0823	2.63	2.88	2.57	2.64	31.95	32.50	31.71	32.47	31.98
Γ	4	16	0.5400	4.32	6.40	4.04	4.70	13.00	16.14	12.73	15.41	13.82
	6	64	0.2130	6.82	9.60	6.54	7.02	22.16	24.22	21.25	23.97	22.67
	8	256	0.0743	9.51	12.81	9.25	9.61	31.89	33.37	30.79	33.28	32.24

Table 1 : Comparison of asymptotic formulas with actual values for Gaussian (G), Laplacian (L), Two-sided Rayleigh (R), and Gamma (Γ) densities.

Observations:

$\hat{L}_{M,1}$ is always too large. It predicts too much distortion (see S_0), even though it ignores overload distortion.

In the Gaussian case, $\Delta_M^2/12$ overerestimates SNR by 0.9 dB at R=4 and by 0.3 dB at R=8. This is mostly due to ignoring overload distortion.

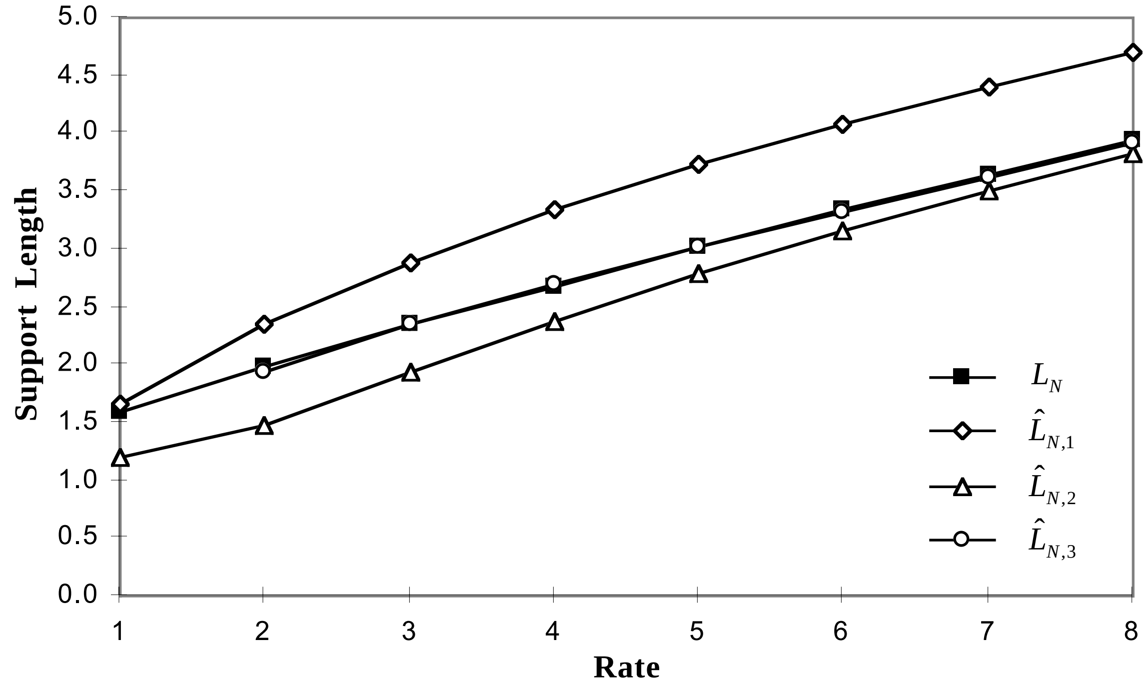

Figure 1 : Optimal Support Length L vs. Rate for USQ and the Gaussian density. (N=M)

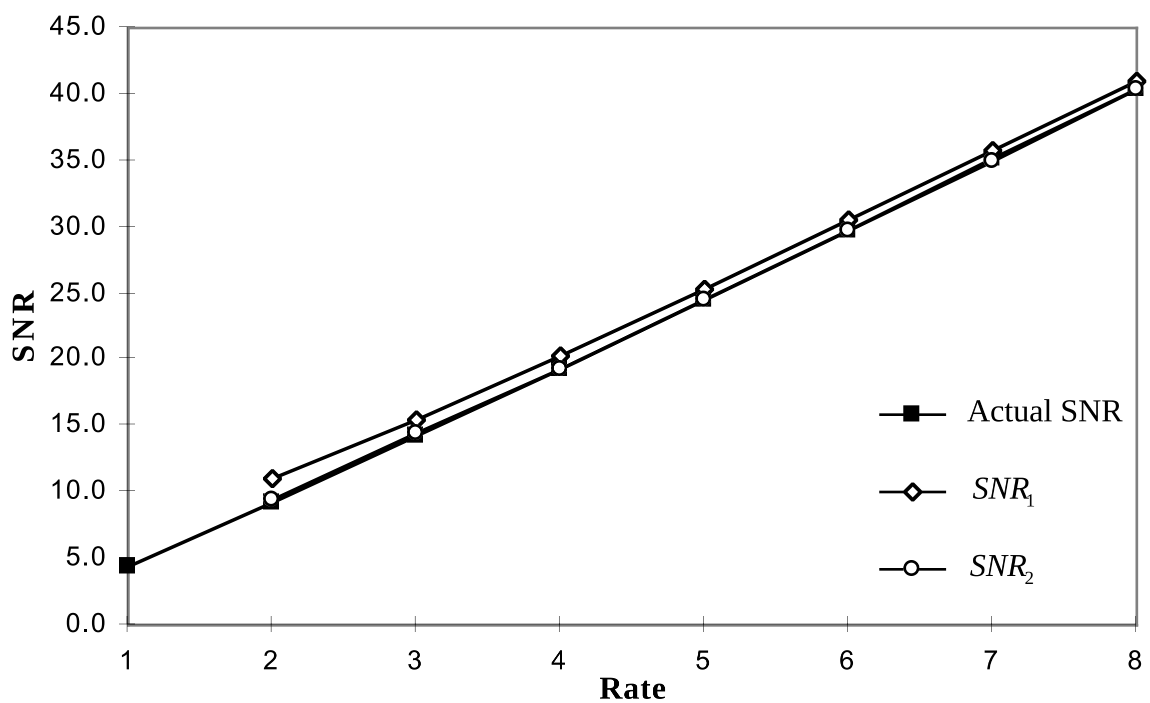

Figure 2 : SNR vs. Rate for USQ optimized for the Gaussian density. (N=M)

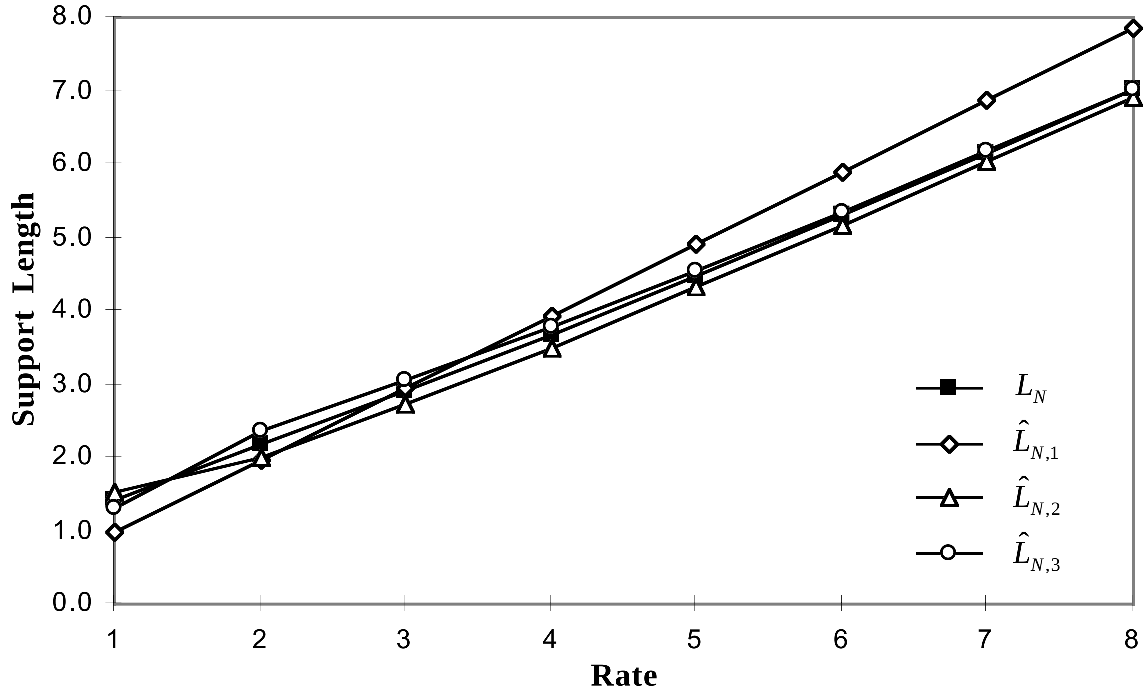

Figure 3 : Optimal Support Length L vs. Rate for USQ and the Laplacian density. (N=M)

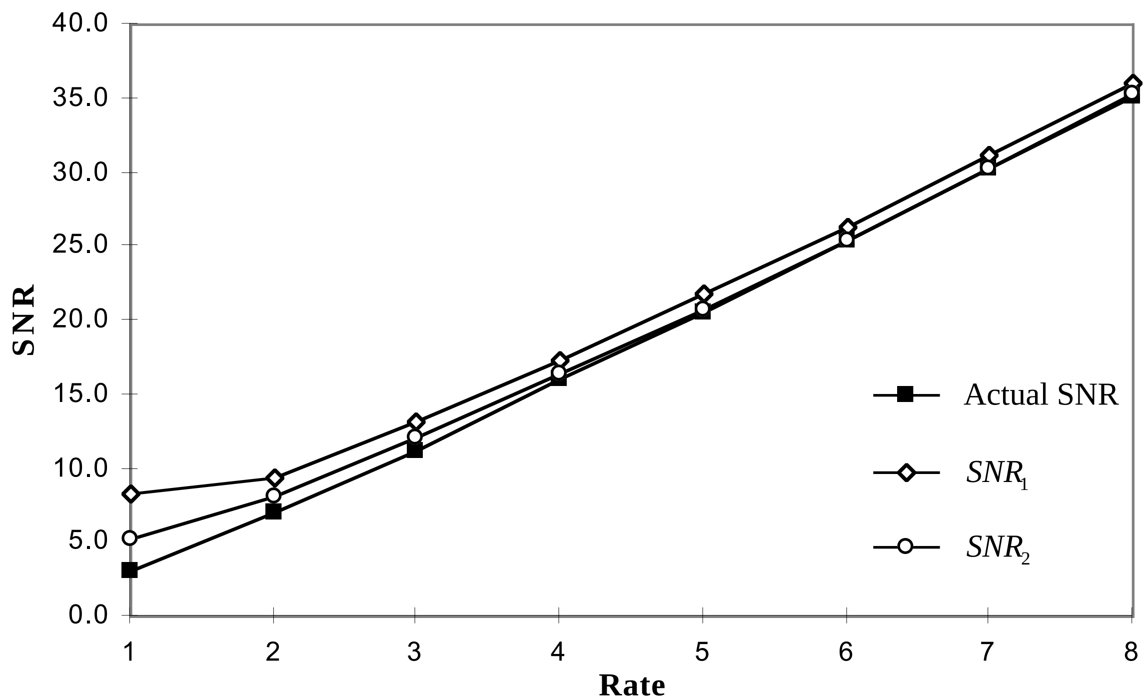

Figure 4 : SNR vs. Rate for USQ optimized for the Laplacian density. (N=M)

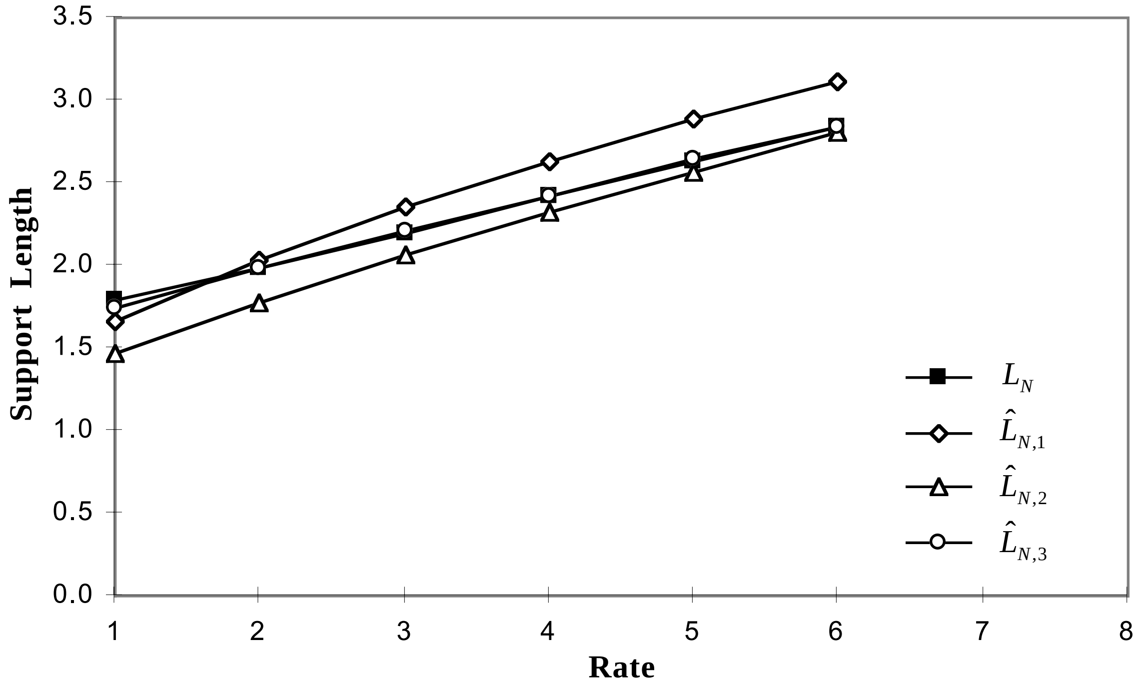

Figure 5 : Optimal Support Length L vs. Rate for USQ and the Rayleigh density. (N=M)

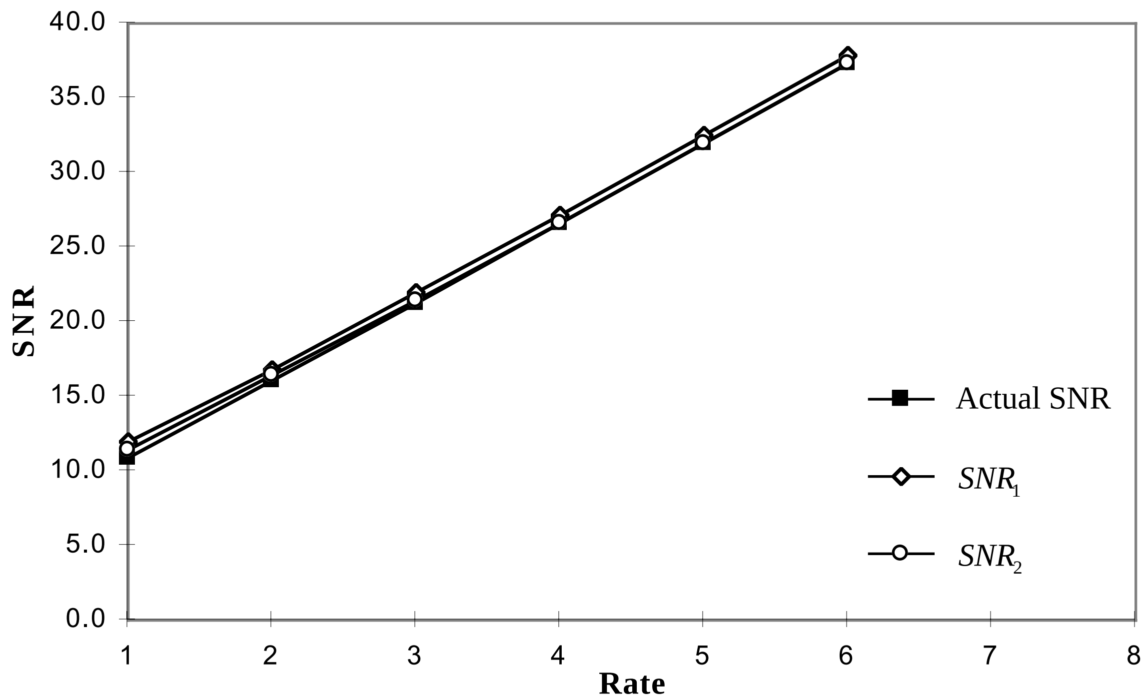

Figure 6 : SNR vs. Rate for USQ optimized for the two-sided Rayleigh density. (N=M)

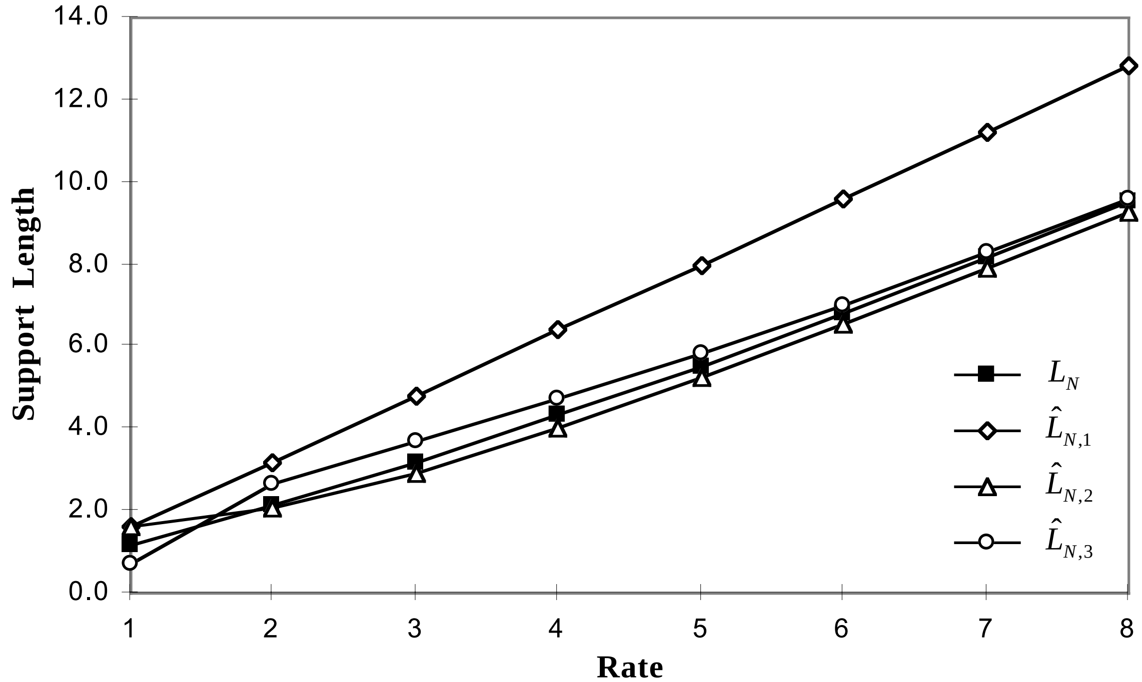

Figure 7 : Optimal Support Length L vs. Rate for USQ and the Gamma density. (N=M)

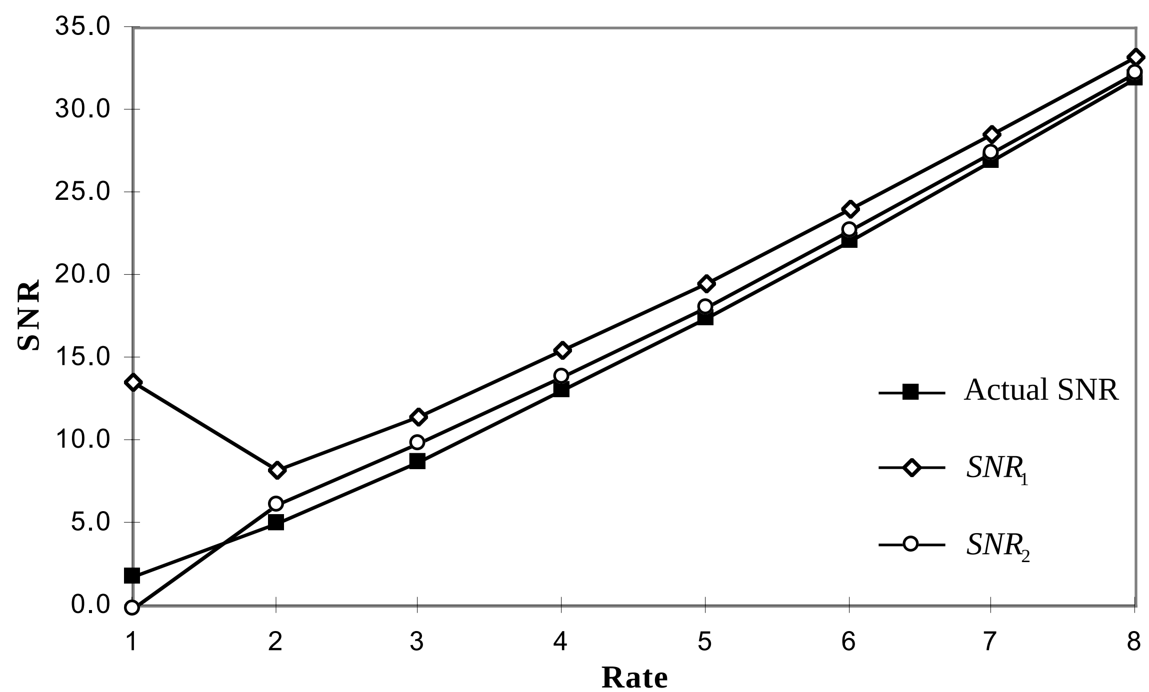

Figure 8 : SNR vs. Rate for USQ optimized for the Gamma density. (N=M)