Homework 2 Solutions **EECS 651** **Jan. 29, 2001**

**** marks lines containing changes relative to the draft version**

1. *Consider three points in two-dimensional space that are not colinear. Show that the perpendicular bisectors between each pair of points meet at one point.*

Consider the triangle formed by the three points a, b and c. The perpendicular bisectors of the sides ab and ac are shown. They meet at a point named d. Since d is on the a perpendicular bisector to ab

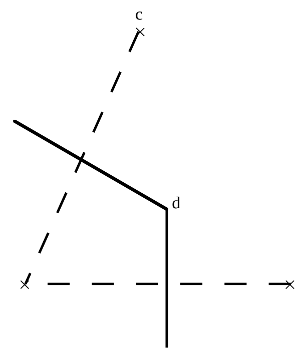
distance(a to d) = distance(b to d)

and since d is on the perpendicular bisector to ac

distance(c to d) = distance(a to d)

Hence,

distance(b to d) = distance(c to d)



Therefore the point d is also on the perpendicular bisector of the side bc. Therefore all three perpendicular bisectors meet at the point d.

This fact implies that the Voronoi regions formed by three codevectors meet at one point.

2. *Suppose $C = \{\underline{w}_1, \underline{w}_2\}$ is a minimum MSE k-dimensional VQ with two codevectors for random vector $\underline{X} = (X_1,\ldots,X_k)$. Show that there exists numbers a and b such that $E\underline{X} = a\,\underline{w}_1 + b\,\underline{w}_2$; $E\underline{X}$ is located on a line between $\underline{w}_1$ and $\underline{w}_2$. Give the values of a and b.*

Since C is a minimum MSE VQ, $\underline{w}_i = E[\underline{X}|\underline{X}\in S_i]$ for i = 1,2. Therefore, by iterated expectation.

** $\mathbf{E\underline{X}} = \Pr(X\in S_1)\, E[\underline{X}|\underline{X}\in S_1| + \Pr(\underline{X}\in S_2)\, E[\underline{X}|\underline{X}\in S_2]$ **=** $\mathbf{a\,E\,w_1 + b\,E\,w_2}$

where **a = Pr($\underline{X}\in S_1$) and b = Pr($\underline{X}\in S_2$)**

3. *A vector quantizer must be designed to quantize at rate 2 bits/sample a source that emits 1000 samples/sec. The device available for encoding can perform 10^6 arithmetic operations per second (floating point or otherwise) and has storage for 100,000 floating point numbers. Find the largest dimension that the VQ can use. (Don't worry about decoding.)*

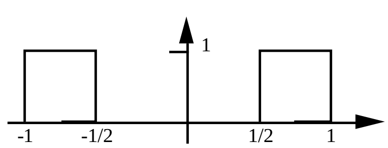
A VQ with dimension k and rate R has $M = 2^{kR}$ quantization vectors. It requires storage for $kM = k2^{kR}$ floating point numbers. Since R = 2, the largest value of k such that $k2^{kR} \le 100{,}000$ is k = 6 .

It also requires M distance calculations per source vector and M-1 comparisons. Each distance calculation requires k substractions, k squarings, and k-1 additions. Thus the total number of operations per source vector is (3k-1)M+M-1 = 3kM-1. Therefore, the processor must be capable of

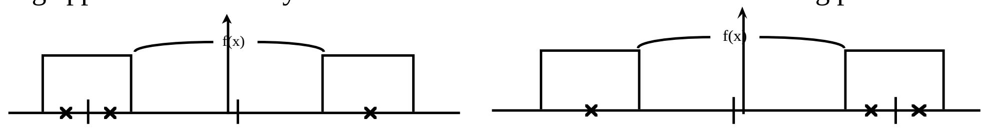
** doing $(3kM\text{-}1) \times \frac{1000}{k}$ operations per second . The largest k such that the above is no larger than 10^6 is k = 4 . (If you used kM or 2kM for the number of operations, you get the same answer.)

Taking both constraints into account, we see that **k = 4 is the largest possible dimension**.

4. (a) *Find as many 3-level scalar quantizers (M=3) as possible that satisfy the two optimality properties for the density $f_X(x)$ shown below.*



The following appear to be the only ones. This didn't turn out to be an interesting problem



One might also consider the quantizer with codebook {-3/4, 0, 3/4} and thresholds {-3/8, 3/8} to satisfy the optimality criteria. Notice that the middle cell has zero probability. Since the centroid of a cell with no probability is something that some might choose to say doesn't exist, in grading this problem, we won't count this quantizer. We'll also ignore other quantizers where one or more cells has zero probability.

(b) *Which of the quantizers is optimal? What is its MSE?*

They both have the same MSE so they are both optimal.

$$\textbf{MSE} = \frac{1}{4}\frac{(1/4)^2}{12} + \frac{1}{4}\frac{(1/4)^2}{12} + \frac{1}{2}\frac{(1/2)^2}{12} = \frac{1}{12}\left(\frac{1}{64} + \frac{1}{64} + \frac{1}{8}\right) = \mathbf{\frac{5}{384}} = \mathbf{0.013}$$

(c) *Find the best 3-level "symmetric" quantizer whose levels and thresholds are symmetric about the origin. (The answer to this should convince you that the best quantizers for symmetric densities are not always symmetric. But what if the number of levels is even. Must the best quantizer with an even number of levels be symmetric for a symmetric density? I'll leave this as an open question for you to think about.)*

This is the quantizer with codebook {-3/4, 0, 3/4} and thresholds {-3/8, 3/8}.

5. *This problem should be done analytically without using a computer.*

(a) *Find a scalar quantizer with three levels that satisfies the optimality critera for the Laplacian pdf*

$$p(x) = \frac{1}{2} e^{-|x|} .$$

If you can find more than one, choose the one with smallest MSE.

Since the density is symmetric, let's try to find a symmetric three-level quantizer. We can't know in advance that the optimum quantizer is symmetric, but it's worth a try. Accordingly, consider a quantizer of the form

```
------×------|-------×-------|------×------
     -w     -t       0       t      w
```

Optimal thresholds: To satisfy the optimality property of thresholds, we must have $t = w/2$, which by symmetry guarantees $-t = -w/2$.

Optimal levels: First note that for the middle cell, $E[X|-t \le X \le t] = 0$ (because the Laplacian density is symmetric), so the centroid property holds. For the rightmost cell, we must have $w = E[X|X \ge t]$, and if this holds then we'll also have for the leftmost cell $-w = E[X|X \le t]$.   Therefore, w must satisfy the equation

$$w = \int_{w/2}^{\infty} x\, p(x|X \ge w/2)\, dx = \int_{w/2}^{\infty} x \frac{p(x)}{\int_{w/2}^{\infty} p(x')\, dx'} dx = \int_{w/2}^{\infty} x \frac{\frac{1}{2} e^{-x}}{\int_{w/2}^{\infty} \frac{1}{2} e^{-x'}\, dx'} dx = \frac{w}{2} + 1,$$

So, $w = 2$ and $t = w/2 = 1$. Thus, the best quantizer has

levels C = { -2, 0, 2} and **thresholds {-1, 1}**.

(b) *Find the resulting MSE. and SNR (in dB).*

$$\mathbf{D} = \int_{-\infty}^{-1} (x+2)^2 \frac{1}{2} e^{-x} dx + \int_{-1}^{1} x^2 \frac{1}{2} e^{-|x|} dx + \int_{1}^{\infty} (x-2)^2 \frac{1}{2} e^{-x} dx = 2 - \frac{4}{e} = \mathbf{.528} .$$

$$\mathbf{SNR} = 10 \log_{10} \frac{\sigma^2}{D} = \mathbf{5.783\ dB}$$

6. *The scalar quantizer shown below is optimum for a random variable X with pfd $f_X(x)$. The probabilities of the three levels are .5, .3 and .2 . The MSE is $D = .2$.*

```
---------×--------------|-------------×---------|--------×--------
        -2             -.5            1         2        3
```

Find EX, $E\,XQ(X)$ and $var(X)$.

In class we found special relationships that hold for quantizers satsifying the centroid condition. We may use these, since an optimal quantizer satisfies the centroid condition.

EX $= E\,Q(X) = .5 \times -2 + .3 \times 1 + .2 \times 3$ = **-.1**

E XQ(X) $= E\,Q^2(X) = .5 \times 4 + .3 \times 1 + .2 \times 9$ = **4.1**

$E\,Q^2(X) = E\,X^2 - E(X-Q(X))^2 = E\,X^2 - D$

$\Rightarrow E\,X^2 = E\,Q^2(X) + D = 4.1 + .2 = 4.3$

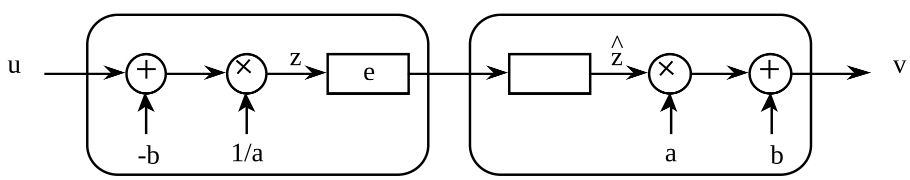
var(X) $= E\,X^2 - (E\,X)^2 = 4.3 - .01$ = **4.29**

7. *In this problem we show how a quantizer designed for a random variable X can be modified to obtain a quantizer for a related random variable. Suppose a scalar quantizer, called Quantizer A, is designed for random variable X, with pdf $f_X(x)$. It has size M, thresholds $t_1,\ldots,t_{M-1}$, levels $w_1,\ldots,w_M$, binary codewords $\underline{c}_1,\ldots,\underline{c}_M$, quantization rule Q, encoding rule e, and decoding rule d. It could be nonuniform, and it does not have to be optimal any sense. And suppose we need a quantizer for random variable U, whose density $f_U(u)$ is related to that of X via*

$$f_U(u) = \frac{1}{|a|} f_X\!\left(\frac{u-b}{a}\right), \text{ where } a \neq 0$$

For example, this would be the case if $U = aX + b$.

Consider the Quantizer B shown below which precedes the encoder e with an addition and a multiplication and follows the decoder d with a multiplication and an addition.



encoder e' of Quantizer B　　　　decoder d' of Quantizer B

(a) *For Quantizer B, find the size M', thresholds $t'_1,\ldots,t'_{M'-1}$, levels $w'_1,\ldots,w'_{M'}$, binary codewords $\underline{c}'_1,\ldots,\underline{c}'_{M'}$, quantization rule Q', encoding rule e', and decoding rule d' in terms of a, b and the corresponding parameters or functions of Quantizer 1. (It might help to draw yourself an example of Quantizer A and Quantizer B.)*

$M' = M$, **$t'_i = at_i + b$, $i = 1,\ldots,M-1$**; **$w'_i = aw_i + b$, $i = 1,\ldots,M$**,

** **binary codewords $\underline{c}'_1,\ldots,\underline{c}'_{M'} = \underline{c}_1,\ldots,\underline{c}_{M'}$**, **$Q'(u) = aQ((u-b)/a) + b$**, **$e'(u) = e((u-b)/a)$**,

$d'(\underline{c}) = a\,d(\underline{c}) + b$, where $\underline{c}$ is the string of bits that the decoder is given.

(b) *Show that the input to e is a random variable Z with the same pdf as X. (You may need to remind yourself of how to find the pdf of one random variable that is a function of another.)*

Since $Z = \frac{u-b}{a}$, then the conventional theory of transformation of random variables shows that

$f_Z(z) = af_U(aZ+b)$

We also know $f_U(u) = \frac{1}{a} f_X(\frac{u-b}{a})$. Substituting $x = \frac{u-b}{a}$ into this implies $f_X(x) = a\, f_U(ax+b)$. And from this we see that f_Z and f_X are identical pdfs.

(c) *Show the MSE, denoted $D_{B,U}$, of Quantizer B operating on U is related to $D_{A,X}$, the MSE of Quantizer A operating on X, via $D_{B,U} = a^2 D_{A,X}$.*

$$D_{B,U} = E\,(U-Q'(U))^2 = E\Big(U-aQ(\tfrac{U-b}{a})\Big)^2 \quad \text{using the formula for } Q' \text{ in terms of } Q$$

$$= E\big(aZ+b - aQ((aZ+b-b)/a) - b\big)^2 \quad \text{since } U = aZ + b$$

$$= E\big(aZ - aQ(Z)\big)^2 = a^2\, E\big(Z-Q(Z)\big)^2 = a^2\, E\big(X-Q(X)\big)^2 \text{ since Z and X have same pdf}$$

$$= D_{A,X}$$

8. *Let C_k and C_m be vq codebooks with rate R and dimensions k and m, respectively. Let $\{X_i\}$ be a stationary source, and let D_k and D_m denote their MSE distortions when used with their respective Voronoi partitions. Let*

$$C = C_k \times C_m = \Big\{(x_1,\dots,x_{k+m}) : (x_1,\dots,x_k) \in C_k \text{ and } (x_{k+1},\dots,x_{k+m})\Big\}$$

C is said to be the product of C_k and C_m. Alternatively, C is said to be a product quantizer.

(a) *Find the dimension, size and rate of the VQ with codebook C.*

** **dimension = k+m, size = $|C_k|\ |C_m|$,**

$$\textbf{rate} = \frac{1}{k+m} \log |C| = \frac{1}{k+m} \log |C_k|\,|C_m| = \frac{k}{k+m}\frac{1}{k} \log |C_k| + \frac{m}{k+m}\frac{1}{m}\log |C_m| = \frac{k}{k+m} R + \frac{m}{k+m} R$$

$$= \mathbf{R}$$

(b) *Show that the MSE distortion D(C) of C, when used with its Voronoi partition on the given source, satisfies*

$$D = \frac{k}{k+m} D_k + \frac{m}{k+m} D_m$$

Let Q_k and Q_m be the quantization rules associated with C_k and C_m and their respective Voronoi partitions; i.e. $Q_k(x_1,\dots,x_k)$ is the closest codevector in C_k to $(x_1,\dots,x_k)$, and similarly for Q_m. Let Q be the quantization rule associated with C and its Voronoi partition, so that $Q(x_1,\dots,x_{k+m})$ is the closest codevector in $C = C_k \times C_m$ to $(x_1,\dots,x_{k+m})$. Clearly, the closest codevector in $C_k \times C_m$ to $(x_1,\dots,x_{k+m})$ consists of the closest codevector in C_k to $(x_1,\dots,x_k)$, followed by the closest codevector in C_m to $(x_{k+1},\dots,x_{k+m})$. Therefore,

$$Q(x_1,\dots,x_{k+m}) = (Q_k(x_1,\dots,x_k), Q_m(x_{k+1},\dots,x_{k+m})), \quad \text{for any } (x_1,\dots,x_{k+m}).$$

Now $$\mathbf{D_{k+m}} = \frac{1}{k+m} E\|\underline{X}-Q(\underline{X})\|^2$$

$$= \frac{1}{k+m} E\|\, (X_1,\dots,X_k,X_{k+1},\dots X_{k+m}) - (Q_k(X_1,\dots,x_k), Q_m(X_{k+1},\dots,X_{k+m}))\,\|^2$$

$$= \frac{1}{k+m} \big(E\|\,(X_1,\dots,X_k) - Q_k(X_1,\dots,X_k)\,\|^2 + E\|\,(X_{k+1},\dots,X_{k+m}) - Q_m(X_{k+1},\dots,X_{k+m})\,\|^2\big)$$

$$= \mathbf{\frac{1}{k+m}\big(k\ D_k + m\ D_m\big)}$$