Homework 3 **EECS 651** **Due: Wed., Jan. 31, 2001**

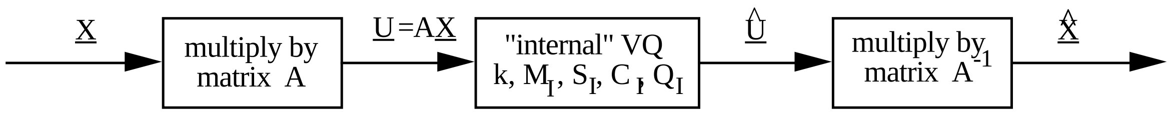
in class, before lecture begins

1. (Problem 11.12, p. 403, from Gersho and Gray). A 2-codeword 4-dimensional VQ is to be designed. The distortion measure is Hamming: d(x,y) = 1 if $x \neq y$, d(x,y) = 0 if x=y. The distortion between vectors is the average of the Hamming distance between their components. Apply the LGB algorithm to the training sequence below

   1111, 1110, 1110, 0001, 1001, 0001, 1000, 0010, 0001, 1101

   Start with initial codebook $C_1 = \{\underline{w}_1, \underline{w}_2\} = \{1100, 0011\}$. Assume that in case of a tie in the distortion between an input vector and two codewords, the training vector is assigned to $\underline{w}_1$. Also assume that in the case of a tie in the centroid computation that a 0 is chosen.

2. Consider the vector quantizer described by the following block diagram. (This is a kind of vector generalization of Problem 7 of the previous homework assignment.)



The source random vector is $\underline{X} = (X_1,...,X_k)^t$ with pdf $p_{\underline{X}}(\underline{x})$. The matrix A is a $k \times k$ orthogonal matrix, which means it has the properites that $A^{-1} = A^t$, the rows are orthonormal, the columns are orthonormal, and $\|A\underline{x}\| = \|\underline{x}\|$ for any $\underline{x}$ (each of these properties implies the others). From the diagram we see that $\underline{U} = A\underline{X}$, $\hat{\underline{U}} = Q_I(\underline{U})$, and $\hat{\underline{X}} = A^{-1}\hat{\underline{U}}$.

(a) Find the codebook C, partition S, quantization rule Q, and rate of the overall quantizer in terms of the matrix A and the corresponding properties of the internal VQ.

(b) Show that the MSE distortion of the overall quantizer operating on $\underline{X}$ equals the distortion of the internal quantizer operating on $\underline{U}$.

(c) Show that if the internal VQ is optimal for $\underline{U}$ (meaning that for its size and dimension it has smallest MSE), then the overall VQ is optimal for $\underline{X}$, regardless of which orthogonal matrix is chosen. (The converse is also true, namely, if the overall VQ is optimal for $\underline{X}$, then the internal is optimal for $\underline{U}$, but you don't have to show it.)

(d) In conventional "transform coding", such as JPEG, a great deal of attention is paid to choosing the orthogonal matrix. Why is this? (Property (b) seems to be saying that it doesn't matter.)

(e) Show that if $\underline{X}$ is multiplied by a constant $1/a > 0$ before being multiplied by the matrix A and if the output of the inverse matrix multiplier is multiplied by a in producing $\hat{\underline{X}}$, then the distortion of the overall quantizer on $\underline{X}$ is a^2 times the distortion of the internal quantizer on $\underline{U}$.

(f) Assuming that the internal quantizer has point density $\lambda_I(\underline{x})$ and inertial profile $m_I(\underline{x})$, find the point density $\lambda(\underline{x})$ and inertial profile $m(\underline{x})$ of the overall quantizer in terms of A and the internal point density and inertial profile.

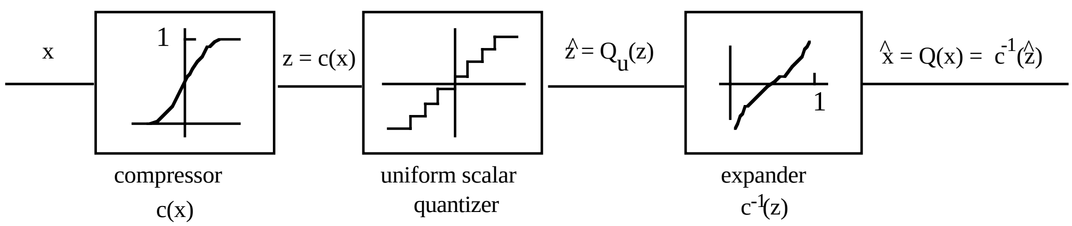
3. Consider a wide-sense stationary, first-order autoregressive source of the form

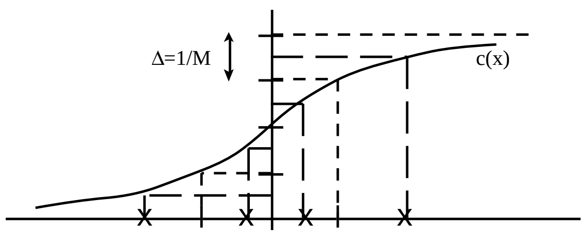
$$X_n = \rho X_{n-1} + Z_n$$

where the Z_n's are IID with zero means and where Z_n is uncorrelated with $X_{n-1}, X_{n-2}, \ldots$. Show that

$$E Z^2 = E X^2 (1-\rho^2) .$$

4. Consider the scalar quantizer shown below, called a *compander*, that quantizes by preceding the encoder of an M level uniform scalar quantizer with support $[0,1]$ with a memoryless nonlinear function $c(x)$. At the decoder, the output of the decoder for the uniform scalar quantizer is followed by the inverse of c. The levels and thresholds of the uniform scalar quantizer are distributed evenly over the interval [0,1]. The function c is nonnegative and montonically increasing, and it maps $(-\infty,\infty)$ into $[0,1]$. The plot below the block diagram may help you to visualize the operation of the compander.





(a) Find formulas for the levels $w_1,\ldots,w_M$ and thresholds $t_0,\ldots,t_M$ of the compander in terms of the function c.

(b) Assuming M is large, find an approximate expression for the distortion of this quantizer in terms of M, the function c, and the probability density of X. Simplify as much as possible (Hint: It should be an integral expression.)

(c) Show that any scalar quantizer can be implemented with a compander, provided its levels lie within its cells.

5. (a) Use Bennett's integral and the results of Parts b and e of Problem 2 to predict the MSE of JPEG applied to the image 'lena' with quality factor 1. To do this you will need to know that JPEG has the form shown in Problem 2 with the internal quantizer consisting of 64 uniform scalar quantizers with step sizes shown in the table that was distributed and posted on the website. The orthonormal transform is preceded by multiplying by $1/a = 16$ and the inverse transform is postmultiplied by $a=16$. (You might want to use Matlab, Excel, or write a computer program to avoid a lot of repititious calculations.)

(b) Compare to the actual distortion of JPEG running on 'lena' with quality factor 1.