Homework 5 Solutions **EECS 651** **Mar. 9, 2001**

1. *A binary IID source has p(0) = .995 and p(1)=.005.*

(a) *Find a fixed-to-variable length block lossless source code with rate no larger than 0.4 . (It should be as simple as possible.)*

"As simple as possible" translates to "with as small a blocklength k as possible". So we start with blocklength k =1 and keep increasing k until we find a code with rate .4 or less.

For k = 1, a best code is {0,1} with rate $R = R_1^* = 1$, which is too large.

For k = 2, the best codes have rate R = .5, which is too large.

For k = 3, the best codes, see for example the code below, have rate R = .33 < .4 so this is the simplest code that meets the specification of the problem.

$u_1\, u_2\, u_3$	$p(u_1,u_2,u_3)$	codeword
000	.985	0
001	.00495	10
010	.00495	110
011	.000241	111100
100	.00495	1110
101	.00241	111101
110	.00241	111110
111	1.25×10^{-7}	111111

(b) *Is it possible to find a fixed-to-variable length block lossless source code with rate less than .1? If so, what can you say about how large its input block length must be?*

Yes, because the source entropy H is .045, which is less than .1.

Its blocklength must be at least 4 because k = 1,2,3 do not give rate less than .1. The source length need not be larger than 19 because $R_{19}^* < H + 1/19 = .0976 < 1$. We don't know if k = 18 will work because $R_{18}^* < H + 1/18 = .1006$. Thus without going to the trouble of following the procedure in part (a), we can say that the smallest source length is at least 4 and no larger than 19.

Moreover, we also know that $\overline{L}$ is always greater than one, and so for any code $R = \overline{L}/k > 1/k$. Thus, if $k \le 10$, the R > .1. Hence, the best bounds to the smallest value k, denoted k_{min} are

$$\mathbf{11 \le k_{min} \le 19}\ .$$

2. *Suppose we are told that a certain stationary source $\{X_k\}$ with alphabet $A = \{1,2,...,M\}$ has*

$$R_3^* = .95 \text{ and } R_4^* = .92$$

where R_k^* denotes the least rate of any block to variable length prefix code with input blocklength k. What can you deduce about the values of H_∞ and H_k for k = 1,2,3, ... ? In other words, using the given information, find the tightest possible bounds to H_∞ and to each H_k .

We need to find upper and lower bounds to each H_k. And we need to make them as tight as we can, given that we know only that $R_3^* = .95$ and $R_4^* = .92$. We can make use of the following facts:

(a) $H_k \le R_k^* < H_k + \frac{1}{k}$ (equivalently, $R_k^* - \frac{1}{k} < H_k \le R_k^*$)

(b) $0 \le H_{k+1} \le H_k \le \log_2 M$ for every k

For k = 3, (a) $\Rightarrow$ $.62 < H_3 \le .95$. For k = 4, (a) $\Rightarrow$ $.67 < H_4 \le .92$

We can also get a tighter lower bound on H_3, since $H_3 \ge H_4 > .67$.
For k =1, we only know $.67 < H_3 \le H_1 \le \log_2 M$. And the same holds for k = 2.

For $k \geq 5$, we only know $0 \leq H_k \leq H_4 \leq .92$.
In summary, the tightest bounds are

$$\mathbf{.67 < H_1 \leq \log_2 M, \quad .67 < H_2 \leq \log_2 M, \quad .67 < H_3 \leq .95, \quad .67 < H_4 \leq .92,}$$

$$\mathbf{0 \leq H_k \leq .92, \quad k \geq 5}$$

3. *Consider the "runlength" code shown below. This is a variable-length to fixed-length code, unlike the fixed-length to variable-length we have considered in class. Let the source be IID source with $p(0) = .995$ and $p(1)=.005$*

source sequence	run length	binary codeword
1	0	000
01	1	001
001	2	010
0001	3	011
00001	4	100
000001	5	101
0000001	6	110
0000000	7	111

(a) Explain why this code is uniquely encodable and decodable.

Let us call the sequences in the left column "source words". Notice that the source words are prefix free. Notice also that every for every infinite binary sequence there is one and only one source word that prefixes it. This means that every infinite source sequence can be parsed in one and only one way into source words. Thus it can be encoded in one and only one way, i.e. it is "uniquely encodable".

The binary codewords are prefix free. Therefore, the code is uniquely decodable in the usual sense.

(b) *Find the average length of the encoded source sequences.*

Let $p = p(0)$. The probability of the source sequence with i 0's followed by a 1 is $p^i(1-p)$, $i = 0,...,6$. The probability of the source sequence with 7 0's is p^7. The average length of the source sequence is

$$\sum_{i=0}^{6} (i+1)\, p^i(1-p) + 7\, p^7 = \mathbf{6.90}$$

(c) Find the rate of this code.

The rate of the code is the ratio of the average number of bits coming from out of the encoder to the averge number entering. Therefore,

$$\mathbf{rate} = \frac{\mathbf{3}}{\mathbf{6.90}} = \mathbf{.435}$$

(d) *Compare the rate and complexity of this code to that found in Problem 1a.*

Rate: The code in Problem 1a has smaller rate.

Complexity: The code in Problem 1a has 8 codewords, which must be stored at both the encoder and decoder. The code in this problem also has 8 codewords. However, they need not be stored because they have an obvious pattern so the storage complexity is essentially zero. The encoder in Problem 1a and the decoder in this problem require no computation. The decoder for the code in Problem 1a must keep checking the received bits to see if they match a codeword. The encoder in this problem need only count the number of zeros until a one occurs. So the encoder in this problem is much simpler than the decoder in Problem 1a. The complexity advantages of runlength coding becomes larger as one demands lower rate, which in Problem 1a is obtained by increasing the blocklength and which in this problem is obtained by increasing the blocklength of the output codewords (equivalently the maximum allowable runlength).

4. Not required.

5. *Assume high rate in this problem.*

(a) *Show that when first-order (n=1) variable-rate coding is applied to scalar quantizers with levels and thresholds that are optimal for fixed-rate coding, then the SNR gain will be two-thirds the SNR gain of optimal SQ-VR (n=1) over optimal SQ-FR.*

First, the SNR gain of SQ-VL over optimal SQ-FL at some rate R is $10 \log_{10} \frac{\beta}{\eta}$,

where $\beta = \frac{1}{\sigma^2}\left(\int_{-\infty}^{\infty} f^{1/3}(x)\ dx\right)^3$ and $\eta = \frac{2^{2h}}{\sigma^2}$.

Now lets find the gain obtained by applying variable-rate coding to a quantizer optimized for fixed-rate coding. A scalar quantizer optimized for FLC has point density $\lambda(x) = f^{1/3}(x) \big/ \int f^{1/3}(x)\ dx$.
When used with VLC it has rate

$$R \cong h + \int_{-\infty}^{\infty} f(x) \log_2 \Lambda(x)\ dx$$

and distortion

$$D \cong \frac{1}{12} \int_{-\infty}^{\infty} f(x)\ \frac{1}{\Lambda^2(x)}\ dx$$

where $\Lambda(x)$ is the unnormalized point density, which is proportional to $\lambda(x)$. So let us assume $\Lambda(x) = c\ f^{1/3}(x)$ where c is a constant that we can vary so that the rate is a desired value. Then substituting this into the expressions for R and D gives

$$R \cong h + \int_{-\infty}^{\infty} f(x) \log_2 c\ f^{1/3}(x)\ dx = h + \int_{-\infty}^{\infty} f(x) \log_2 c\ dx + \int_{-\infty}^{\infty} f(x) \log_2 f^{1/3}(x)\ dx$$

$$= -\int_{-\infty}^{\infty} f(x) \log_2 f(x)\ dx + \log_2 c + \frac{1}{3} \int_{-\infty}^{\infty} f(x) \log_2 f(x)\ dx = \frac{2}{3} h + \log_2 c$$

and

$$D \cong \frac{1}{12} \int_{-\infty}^{\infty} f(x)\ \frac{1}{c^2 f^{2/3}(x)}\ dx = \frac{1}{c^2}\frac{1}{12} \int_{-\infty}^{\infty} f^{1/3}(x)\ dx = \frac{\sigma^{2/3}\beta^{1/3}}{12c^2}$$

From the expression for R we have $c = 2^{R-\frac{2}{3}h}$ and substituting this into the expression for D gives an expression for D vs. R:

$$D \cong \frac{\sigma^{2/3}\beta^{1/3}}{12\ 2^{2R-\frac{4}{3}h}} = \frac{1}{12}\ \sigma^{2/3}\beta^{1/3}\ 2^{\frac{4}{3}h}\ 2^{-2R} = \frac{1}{12}\ \sigma^{2/3}\beta^{1/3}\ (\sigma^2\eta)^{2/3}\ 2^{-2R} = \frac{1}{12}\ \sigma^2\beta^{1/3}\eta^{2/3}\ 2^{-2R}$$

Therefore the gain over optimal SQ-FL is

$$\frac{\frac{1}{12}\ \sigma^2\beta\ 2^{-2R}}{\frac{1}{12}\ \sigma^2\beta^{1/3}\eta^{2/3}\ 2^{-2R}} = \left(\frac{\beta}{\eta}\right)^{2/3} \quad \text{or} \quad 10 \log_{10} \left(\frac{\beta}{\eta}\right)^{2/3} = \frac{2}{3}\ \log_{10} \frac{\beta}{\eta}\ \text{ dB}$$

which is 2/3's of the gain of optimal SQ-VL over optimal SQ-FL.

(b) *For Gaussian and Laplacian densities, make a table of the SNR gains of VR coding applied to optimal FR quantizers over the performance of optimal FR quantizers, and also for the gains of optimal SQ-VR (n=1) over optimal SQ-FR.*

Table of gains	VLC of opt'l FL-SQ over FLC of opt'l FL-SQ	opt'l SQ-VL over opt'l SQ-VL
Gaussian	1.87 dB	2.81 dB
Laplacian	3.75 dB	5.63 dB

6. (a) *Find a high-resolution expression for the probability of the cell containing* $\underline{x}$ *assuming the quantizer is optimized for variable-rate coding and qualitatively compare the expression to that for fixed-rate coding.*

If the quantizer is optimized for variable-rate coding, then the point density is a constant, namely,

$$\Lambda(\underline{x}) = \Lambda_k^* = 2^{k(R-h_{kn})}$$

This means that all cells have essentially the same volume. Therefore, the probability of the cell $S_{\underline{x}}$ containing $\underline{x}$ is proportional to $p(\underline{x})$. Specifically, $|S_{\underline{x}}| \cong 2^{-k(R-h_{kn})}$ and

$$\Pr(S_{\underline{x}}) \cong 2^{-k(R-h_{kn})}\, p(\underline{x})$$

In comparison for fixed-rate coding,

$$\Pr(S_{\underline{x}}) \cong p(\underline{x})\, |S_{\underline{x}}| \cong p(\underline{x}) \frac{c}{M}\, p(\underline{x})^{-k/(k+2)} = \frac{c}{M}\, p(\underline{x})^{2/(k+2)}$$

Since the exponent $2/(k+2)$ for fixed-rate coding is less than that for ariable-rate coding, the cell probabilities for fixed-length coding tend to be more nearly equal than for variable-rate coding. This causes the entropy of quantizer designed for variable-rate coding to be smaller than the entropy of the quantizer designed for fixed-rate coding.

(b) *Repeat the above with "probability" replaced by "distortion contribution".*

The distortion contribution of the the cell $S_{\underline{x}}$ for quantizer optimized for variable-rate coding is:

$$\frac{1}{k}\int_{S_{\underline{x}}} ||\underline{x}-Q(\underline{x}')||^2\, p(\underline{x}')\, d\underline{x}' \cong \frac{1}{k}\, p(\underline{x}) \int_{S_{\underline{x}}} ||\underline{x}'-Q(\underline{x}')||^2\, d\underline{x}'$$

$$= \frac{1}{k}\, p(\underline{x})\, k\, m_k^*\, |S_{\underline{x}}|^{(k+2)/k} = p(\underline{x})\, m_k^* \left(2^{-k(R-h_{kn})}\right)^{(k+2)/k} = p(\underline{x})\, m_k^*\, 2^{-(k+2)(R-h_{kn})}$$

In comparison for the quantizer optimized for fixed-rate coding

$$\frac{1}{k}\int_{S_{\underline{x}}} ||\underline{x}-Q(\underline{x}')||^2\, p(\underline{x}')\, d\underline{x}' \cong p(\underline{x})^{-2/(k+2)}\, m_k^*.$$

Whereas for the quantizer optimized for fixed-rate coding, all cells contribute equally to the distortion, for the quantizer optimized for variable-rate coding, the contribution to distortion is proportional to $p(\underline{x})$.

7. *Show that for scalar quantizers optimized for variable-rate coding, the levels should be centroids but the thresholds need not be halfway between the levels.*

Given a set of thresholds, the choice of levels affects the distortion but not the rate. Therefore, in an optimal quantize the levels must be chosen to minimize distortion, which we know means they should be centroids.

We use a counterexample to show that the thresholds need not be halfway between the levels. Consider a quantizer optimized for quantizing an exponential density $p(x) = e^{-x}$, $x \geq 0$, with variable-rate coding . Let the rate be large. We know that the quantizer has approximately a constant point denstiy. Thus the cells should all have the approximately the same width, say Δ. We also know the levels will be the centroids. Since the density is exponential the centroid will be in the same relative position within each cell and this position is to the left of the cell center. Let's say it is at distance δ from the left boundary of the cell, where $0 < \delta < \Delta/2$. It is now evident that each threshold is at distance δ from the next level and distance $\Delta-\delta$ from the previous level. Since $\delta \neq \Delta/2$, the threshold is not halfway between two levels.