Homework 6 Solutions ***DRAFT*** **EECS 651** **March 20, 2001**

1. *Let $\underline{X} = (X_1,...,X_k)$ be a real-valued zero-mean, random vector with covariance matrix $K = [K_{i,j}]$. Show that if K is singular, then there exists some i such that X_i is a linear combination of the other random variables. Hint: Let $\underline{K}_j$ denote the jth row of K. Show that if $\underline{K}_i = \sum_{j\neq i} b_j \underline{K}_j$ for some i and some choice of coefficients b_j, $j = 1,...,k, j\neq i$, then $E[X_i - \sum_{j\neq i} b_j X_j]^2 = 0$.*

   Since K is singular there exists some row $\underline{K}_i$ and some coefficients $b_1, b_2, \ldots$ such that $\underline{K}_i = \sum_{j\neq i} b_j \underline{K}_j$. For future reference, note that

$$K_{i,i} = \sum_{j\neq i} b_j K_{j,i} \quad \text{and} \quad K_{i,k} = \sum_{j\neq i} b_j K_{j,k}.$$

   Now,

$$
\begin{aligned}
E\Big(X_i - \sum_{j\neq i} b_j X_j\Big)^2 &= E\Big(X_i - \sum_{j\neq i} b_j X_j\Big)\Big(X_i - \sum_{k\neq i} b_k X_k\Big) \\
&= E X_i^2 - 2\sum_{j\neq i} b_j E X_i X_j + \sum_{k\neq i} b_k \sum_{j\neq i} b_j E X_j X_k \\
&= K_{i,i} - 2\sum_{j\neq i} b_j K_{j,i} + \sum_{k\neq i} b_k \sum_{j\neq i} b_j K_{j,k} \\
&= K_{i,i} - 2K_{ii} + \sum_{k\neq i} b_k K_{i,k} = K_{i,i} - 2K_{i,i} + K_{i,i} = 0
\end{aligned}
$$

   Postscript to problem: If $E(X-Y)^2 = 0$, then Chebychev's inequality implies

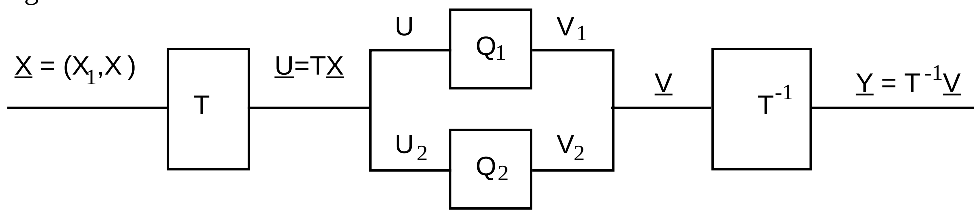
$$\Pr(|X-Y| > \varepsilon) \le \frac{E(X-Y)^2}{\varepsilon} = 0 \quad \text{for any } \varepsilon > 0.$$

   Hence, $\Pr(X \neq Y) = 0$, so X equals Y (Recall the definition of equality of random variables.)

   Applying this to our problem gives $X_i = \sum_{j\neq i} b_j X_j$; i.e. X_i is a linear combination of X_i's.

2. (a) *Design a 2-dimensional (fixed-rate) transform code with rate $R = 1$ bit/sample for a first-order Gaussian autoregressive source with variance 1 and correlation coefficient $\rho = .9$. The code should have as small MSE as possible. (You'll have to specify the transform and the scalar quantizers, as well as give an overall block diagram.) Do not use high resolution analysis. But you will need to use the scalar quantization tables distributed in class and posted on the website.*

Block diagram



Transform T: As shown in class, for large rates, the best matrix is the Karhunen-Loeve Transform (KLT). Rate 1 is not large, but we'll use the KLT anyway, because its all that we know. Its columns are an orthonormal set of eigenvectors for the covariance matrix of $\underline{X} = (X_1,X_2)$.
Since the source is AR with mean zero and correlation coefficient $a = .9$, the covariance matrix is (assuming the source has unit variance) $K_{\underline{X}} = \begin{bmatrix} 1 & .9 \\ .9 & 1 \end{bmatrix}$, its eigenvalues are $\lambda_1 = 1.9$ and $\lambda_2 = .1$, and a pair of corresponding orthonormal eigenvectors are $\underline{w}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\underline{w}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Thus,

$\mathbf{T} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. This makes U_1 have variance $\sigma_1^2 = \lambda_1 = 1.9$ and U_2 have variance $\sigma_2^2 = \lambda_2 = .1$.

Scalar Quantizers Q_1 and Q_2: Q_1 and Q_2 will be optimal fixed-rate quantizers for U_1 and U_2 with rates R_1 and R_2, which must themselves be determined. For a given choice of R_1 and R_2, the distortion will be $D = (D_1+D_2)/2$ where $D_1 = \sigma_1^2\delta_{g,sq\text{-}fl}(R_1)$ and $D_2 = \sigma_2^2\,\delta_{g,sq\text{-}fl}(R_2)$ are the distortions of Q_1 and Q_2, and $\delta_{g,sq\text{-}fl}(R)$ is the OPTA function for fixed-rate scalar quantization for a unit variance Gaussian variable. Specifically, from the Jayant and Noll tables we have

R	0	1	2	3
$\delta_{g,sq\text{-}fl}(R)$	1	.363	.117	.0345

We also know $(R_1+R_2)/2 = 1$. The (reasonable) possible choices for R_1 and R_2 and the resulting distortions are shown below.

R_1	R_2	D_1	D_2	$D=(D_1+D_2)/2$
0	2	1.9	.0117	.956
1	1	.690	.0363	.363
2	0	.222	.1	.161

We see that $R_1 = 2$, $R_2 = 0$ are best. Then from the Jayant and Noll tables we find

Q_1 has levels $(-1.51, -.453, .453, 1.51) \times \sigma_1$ **= (-2.08, -.641, .641, 2.08)**

and **thresholds** $(-\infty, -.982, 0, .982, \infty) \times \sigma_2$ **= $(-\infty$, -.311, 0, .311, $\infty)$**

(b) *Compute the MSE of this code.*

As shown above **MSE = .161**.

(c) *Compare the MSE to that of an optimal scalar quantizer with rate 1.*

The MSE of an optimal quantizer for a unit variance Gaussian source is .**363**. Thus transform coding has **reduced the MSE by the factor 2.25 or 3.53 dB**.

3. (a) *Estimate how much more music (in percent) an audio CD could store if it used fixed-rate transform coding. There is no limit on the dimension of the transform.*

The conventional scheme with which to compare is described in Problem 1a of HW 4. The transform code must attain the same SNR.

Assume the source is the one described in Problem 1 of HW 4.

The conventional scheme from Problem 1a of HW 4 has rate R = 16 and SNR = 81.1 dB.
For optimized k-dimensional fixed-rate transform coding for a Gaussian source,

$$S_{transform}(k,R) = 6.02\ R + 10 \log_{10} \frac{12}{\beta} + 10 \log_{10} \frac{\sigma^2}{|K^{(k)}|^{1/k}}$$

This increases with k to the limit

$$S_{transform}(R) = 6.02\ R + 10 \log_{10} \frac{12}{\beta} + 10 \log_{10} \frac{\sigma^2}{Q}$$

where $\beta = 32.6$ and Q = one-step prediction error = $\sigma^2 (1-\rho^2) = .19\ \sigma^2$. Therefore,

$$S_{transform}(R) = 99.2 \text{ dB}.$$

This is a gain of 18.1 dB. Since transform coding changes 6 dB for every change in the rate we could attain SNR = 88.1 dB, with rate R = 16 - 18.1/6.02 = 13.0. If T is the number of bits a CD can hold, then storage has increased from T/16 to T/13. As a fraction the increase is

$$\frac{T/13.0 - T/16}{T/16} = \frac{16}{13.0} - 1 = .23, \text{ i.e. } \mathbf{23\%}$$

(b) *What is the smallest dimension k such that transform coding with this dimension and the rate found in (a) comes with .5 dB of the desired SNR?*

By examining the formula for $S_{transform}(k,R)$, we see that we need to find the smallest such that

$$10 \log_{10} \frac{|K^{(k)}|^{1/k}}{Q} \leq .5 \text{ dB}$$

For a first-order AR source $|K^{(k)}|^{1/k} = \sigma^2 (1-\rho^2)^{(k-1)/k}$ and $Q = \sigma^2(1-\rho^2)$. Thus we need to find the smallest k such that

$$10 \log_{10} (1-\rho^2)^{-1/k} \leq .5 \quad \text{or} \quad -\frac{1}{k}\, 10 \log_{10} .19 \leq .5 \quad \text{or} \quad k \geq 14.4.$$

Thus **the smallest k is 15.**