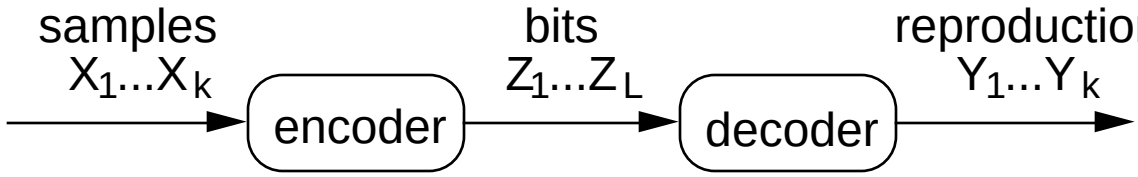
VECTOR QUANTIZATION (VQ)

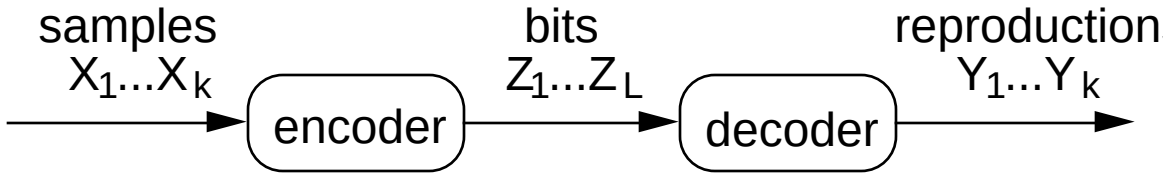
Introduction

Consider an arbitrary fixed-rate lossy source code that operates independently on blocks (vectors) of k real-valued samples: k samples into encoder, L bits out



- source coder = encoder + decoder
- encoder and decoder are described by functions called encoding rule and decoding rule, respectively
- the sets of all possible encoder outputs and all possible decoder outputs play important roles
- the partition induced on space of k-dimensional input vectors plays an important role
- a lossy source code that operates independently on fixed-length blocks, producing fixed-length blocks of bits is called a

  fixed-rate (memoryless) vector quantizer (VQ)
- fixed-rate VQ is a very general paradigm that includes many lossy source codes as special cases, e.g. fixed-rate transform coding. Since it is quite generable and also *analyzable*, it provides an excellent framework for studying lossy source codes.
- JPEG has variable, not fixed, rate. Except for the encoding of dc coefficients, it operates independently on bocks of 64 pixels. (It's a variable-rate VQ with memory.)



KEY CHARACTERISTICS (high-level, input-output)

Dimension: k

Encoding rule: e: $R^k \rightarrow \{0,1\}^L$

$Z_1 \ldots Z_L = e(X_1 \ldots X_k)$,

$Z_{L+1} \ldots Z_{2L} = e(X_{k+1} \ldots X_{2k})$,

$Z_{2L+1} \ldots Z_{3L} = e(X_{2k+1} \ldots X_{3k})$, ...

Binary codebook:

$C_b = \{e(\underline{x}): \underline{x} \in R^k\} = \{\underline{c}_1, \underline{c}_2, \ldots, \underline{c}_M\}$

where $\underline{c}_i = (c_{i1}, c_{i2}, \ldots, c_{iL})$

= ith binary codeword

Size of code: M

Decoding rule d: $C_b \rightarrow R^k$

$Y_1 \ldots Y_k = d(Z_1 \ldots Z_L)$,

$Y_{k+1} \ldots Y_{2k} = d(Z_{L+1} \ldots Z_{2L})$,

$Y_{2k+1} \ldots Y_{3k} = d(Z_{2L+1} \ldots Z_{3L})$, ...

(Quantization) Codebook:

$C = \{d(\underline{c}_1), d(\underline{c}_2), \ldots, d(\underline{c}_M)\}$

$= \{\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_M\}$

where $\underline{w}_i = (w_{i1}, \ldots, w_{ik})$ = ith codevector

(code/reproduction, vector/point)

Quantization rule: Q: $R^k \rightarrow R^k$

$Q(\underline{x}) = d(e(\underline{x}))$ = reproduction produced by decoder for $\underline{x}$

(Quantizing) Partition: $S = \{S_1, S_2, \ldots, S_M\}$

where $S_i = \{\underline{x} \in R^k: e(\underline{x}) = \underline{c}_i\}$

$= \{\underline{x} \in R^k: Q(\underline{x}) = \underline{w}_i\}$

= ith (quantizing) cell

(A *partition* of R^k is a collection of disjoint subsets of R^k whose union is R^k. The elements of the collection are called *cells*.)

Summary: A VQ is characterized by

Dimension: k

Size: M

Encoding rule: e

characterized by

Partition: $S = \{S_1, S_2, \ldots, S_M\}$ and

Binary codebook: $C_b = \{\underline{c}_1, \underline{c}_2, \ldots, \underline{c}_M\}$

Decoding rule: d

characterized by

Codebook $C = \{\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_M\}$

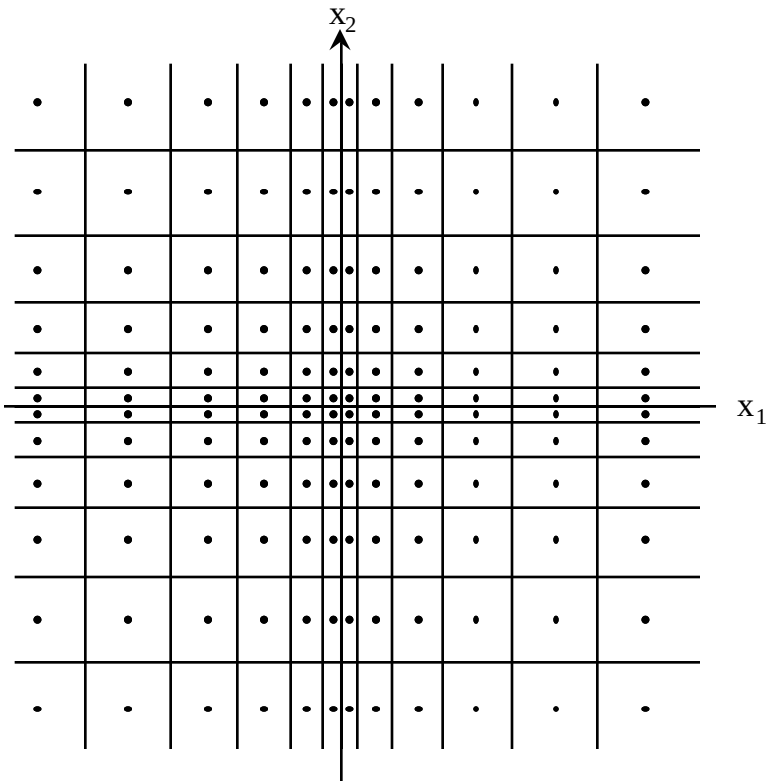
Quantization rule: Q

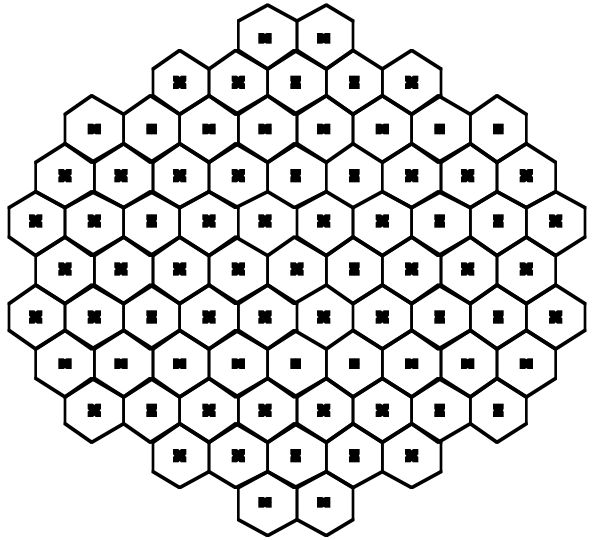
characterized by S and C

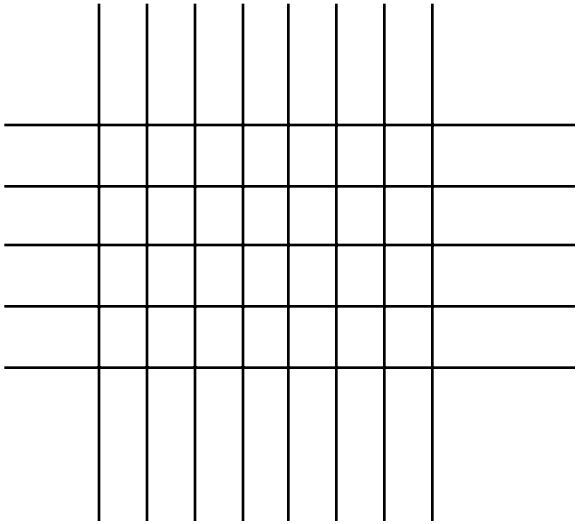
$Q(\underline{x}) = \underline{w}_i$ when $\underline{x} \in S_i$

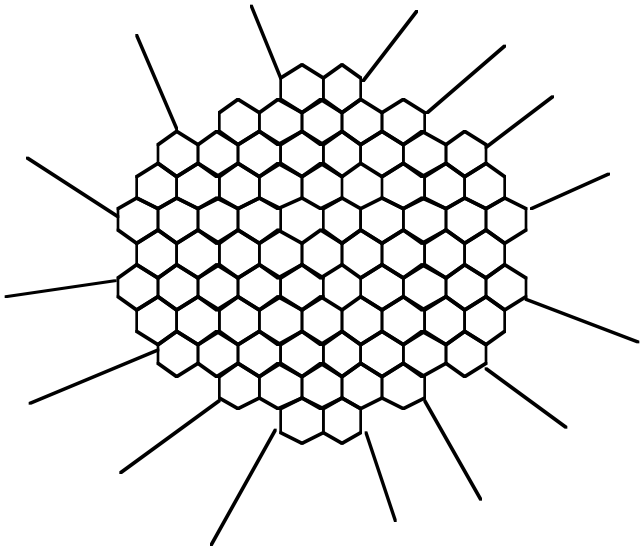
(You must learn to use this terminology and notation.)

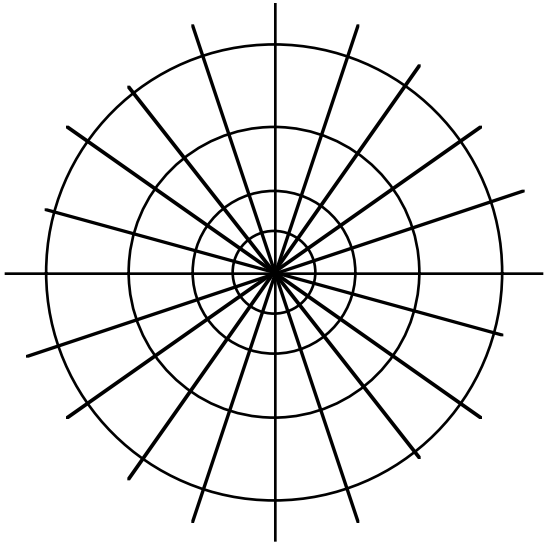
Examples (in k=2 dimensions)

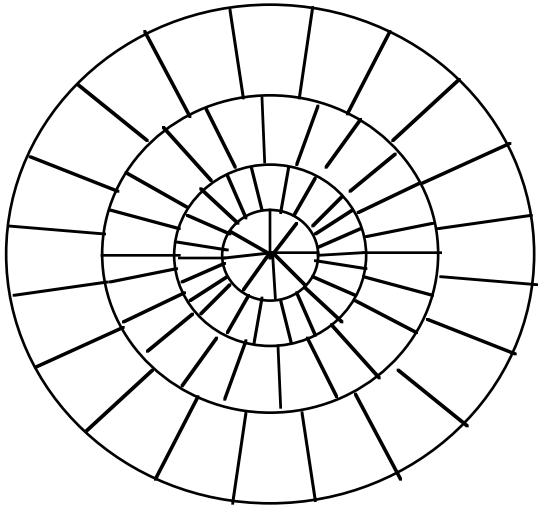


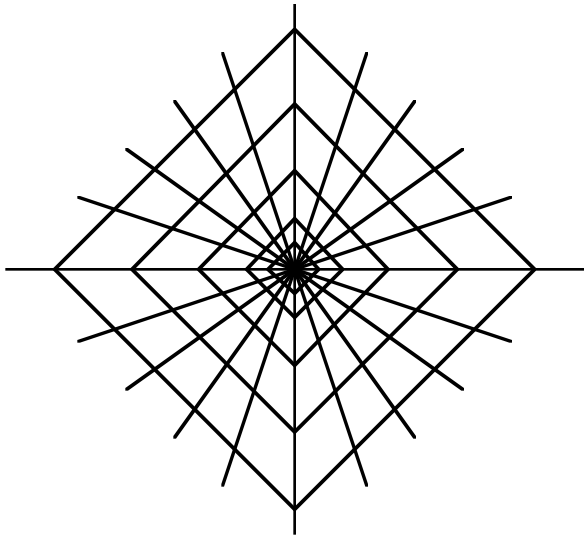


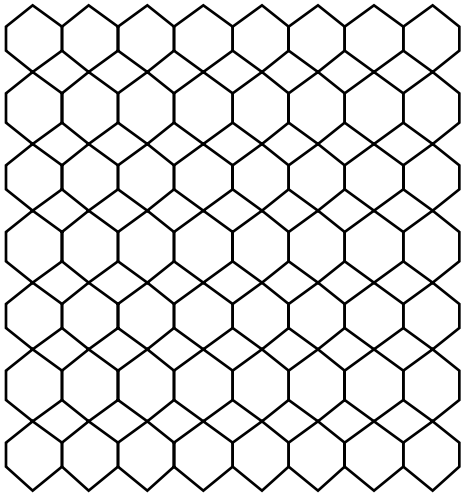


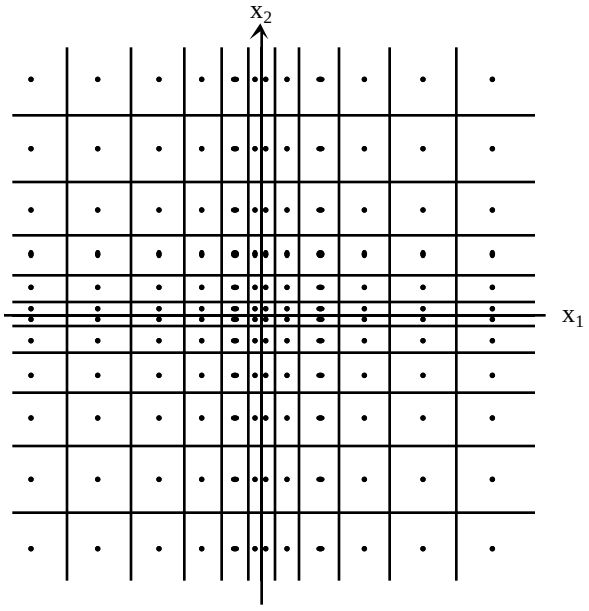


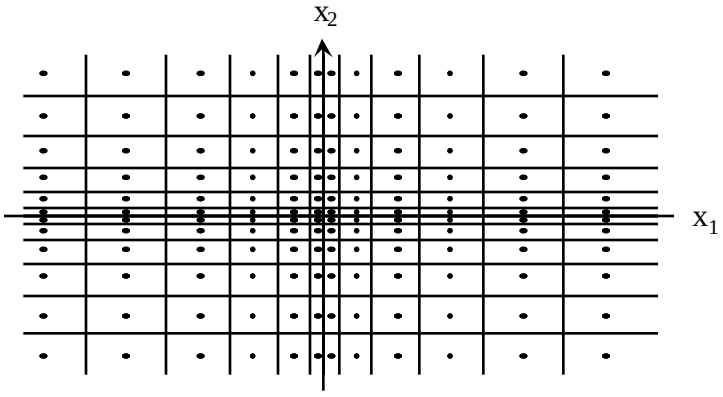









x2
x1


x2
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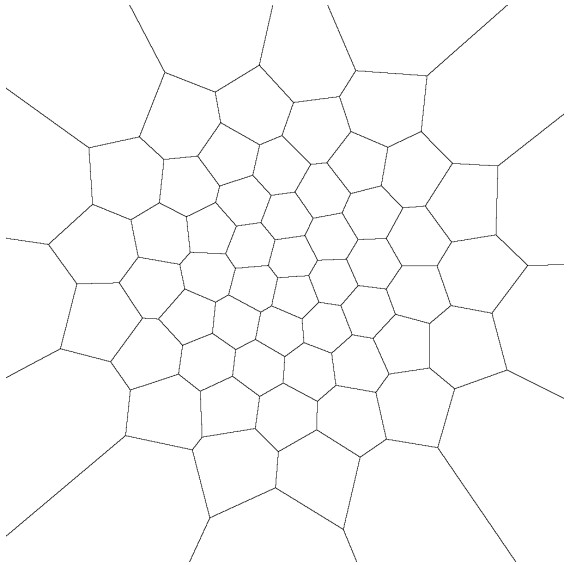

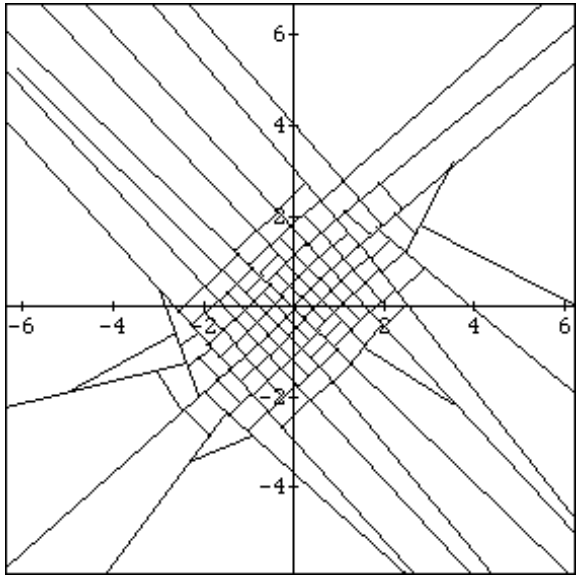
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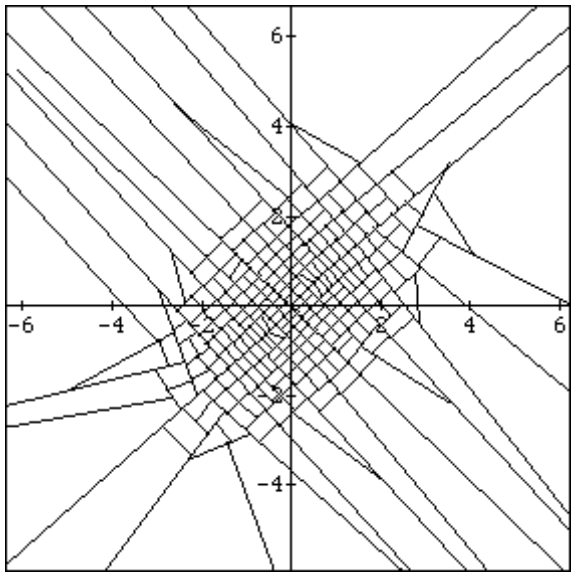
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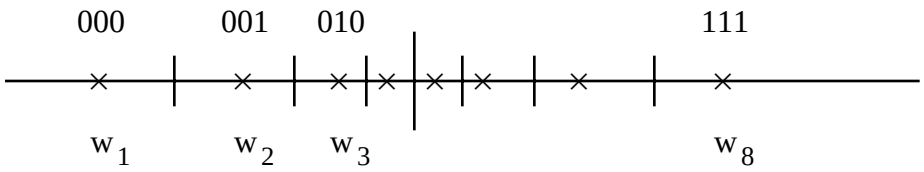
Scalar Quantizer (k=1)



PERFORMANCE

RATE

$$R = \frac{L}{k}$$

$$= \begin{cases} \frac{1}{k}\log_2 M, & \text{if } M=\text{power of } 2 \\ \frac{1}{k}\lceil \log_2 M \rceil, & \text{if } M\neq\text{power of } 2 \text{ \& one index coded at a time} \\ \frac{1}{mk}\lceil \log_2 M^m \rceil \cong \frac{1}{k}\log_2 M, & \text{if } M\neq\text{power of } 2 \text{ \& m indices coded at a time} \end{cases}$$

Accordingly, we assume

$R = \frac{1}{k}\log_2 M$ unless there is need to be picky.

Units: bits/sample

Note: Rate is determined by the encoder, not the decoder.

DISTORTION

D = MSE = mean squared error (normalized by dimension)

$$= \frac{1}{k}\sum_{i=1}^{k} E(X_i-Y_i)^2 = \frac{1}{k} E\sum_{i=1}^{k} (X_i-Y_i)^2$$

$$= \frac{1}{k} E\,||\underline{X}-\underline{Y}||^2 = \frac{1}{k} E\,||\underline{X}-Q(\underline{X})||^2 = \frac{1}{k}\int ||\underline{x}-Q(\underline{x})||^2 f_{\underline{X}}(\underline{x})\, d\underline{x}$$

$$= \frac{1}{k}\sum_{i=1}^{M} \int_{S_i} ||\underline{x}-\underline{w}_i||^2 f_{\underline{X}}(\underline{x})\, d\underline{x}$$

where

$||\underline{x}-\underline{y}||$ = Euclidean distance = $\sqrt{\sum_{i=1}^{k} (x_i-y_i)^2}$

expected value is with respect to probability distribution on $\underline{X} = (X_1,...,X_k)$

Source vector

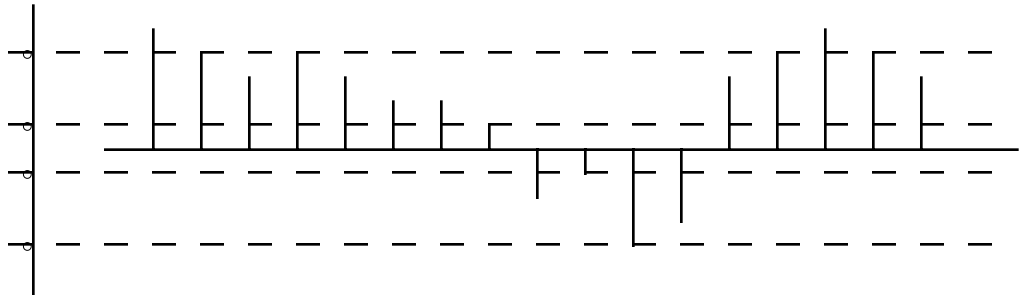
We assume $\underline{X}$ is modeled as a vector of continuous random variables $\underline{X} = (X_1,...,X_k)$, whose probability distribution is characterized by a joint density denoted $f_{\underline{X}}(\underline{x})$ (or just $f(\underline{x})$).

When a VQ operates on sequence of vectors, $\underline{X}_1,\underline{X}_2,...$, we usually assume these vectors come from a stationary random process.

Note: Distortion is determined by Q (or partition and codebook). The specific encoding rule, decoding rule and binary codebook do not matter, except as they determine Q.

HOW VQ DIFFERS FROM SQ

Consider output of correlated source



Consider scalar quantization with four levels: $C_1 = \{-4,-1,1,4\}$.

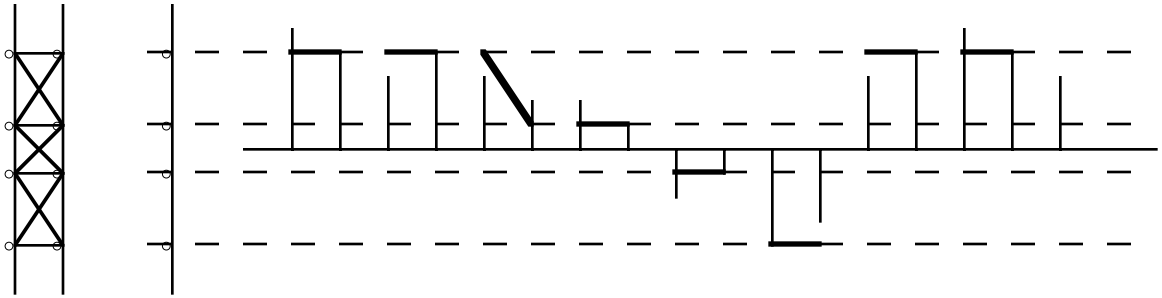
Since successive source samples are mostly quantized into adjacent levels, consider the following 2-dimensional VQ, with only 10 pairs of levels (codevectors)

$C_2 = \{(-4,-4),(-4,-1),(-1,-4),(-1,-1),(-1,1),(1,-1),(1,1),(1,4),(4,1),(4,4)\}$

Rather than the 16 pairs produced by scalar quantization,

$C_1 \times C_1 = \{(-4,-4),(-4,-1),(-4,1),(-4,4),(-1,-4),\ldots,(4,4)\}$

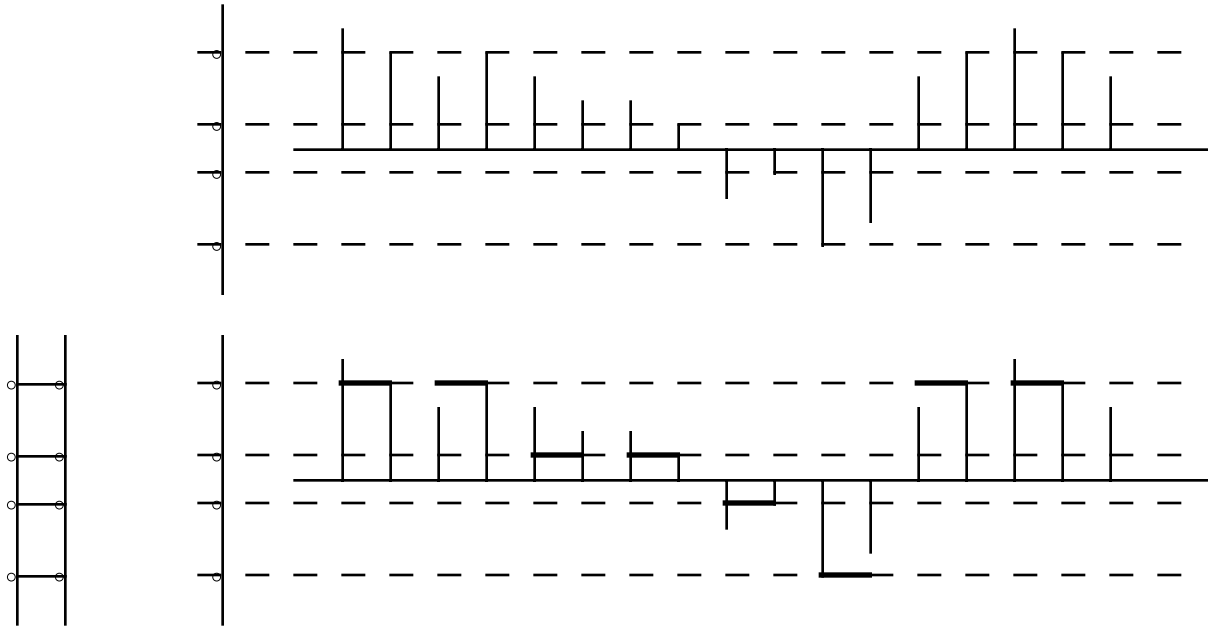
VQ output



Or consider the following 2-dimensional VQ, with codebook consisting only of 4 pairs of levels (codevectors),

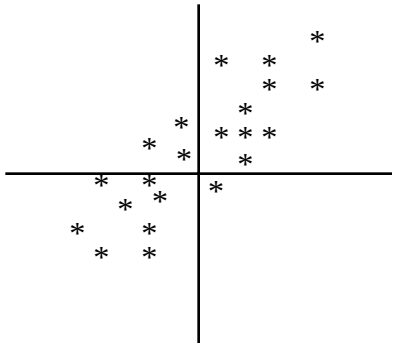
$$C_2 = \{(-4,-4),(-1,-1),(1,1),(4,4)\}$$

rather than the 16 pairs produced by scalar quantization,

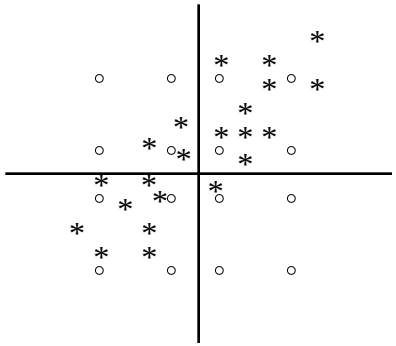


Another view

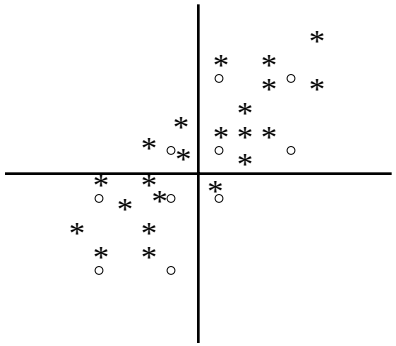
Scatter plot of typical (x_1,x_2) pairs



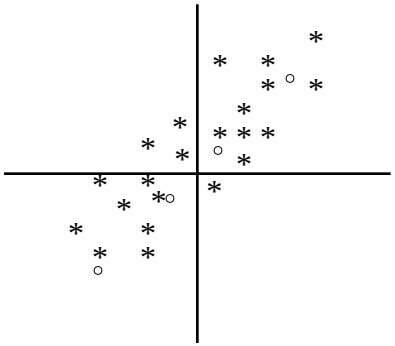
scalar quant. used twice



VQ with M = 10



VQ with M = 4



KEY QUESTIONS:

- How to implement the encoder, i.e. the partitioning?
- Complexity?
- How to design/optimize fixed-rate VQ's?

  (What properties do good fixed-rate VQ's have?)
- How to estimate MSE of a VQ?
- How to design low complexity VQ's with good performance?
- What is best possible performance (D vs. R) of a VQ? (the opta function) How does it depend on dimension k?
- How well do low complexity VQ's perform? What is it in their structure that limits their performance?

OUTLINE OF COVERAGE

- Optimality properties of fixed-rate VQ's.
- "Full search" encoding.
- Generalized Lloyd iterative VQ design algorithms
- Properties of optimal quantizers,

  e.g. $E\,||\underline{Y}||^2 = E\,||\underline{X}||^2 - E\,||\underline{X}-\underline{Y}||^2$
- High-Resolution Analysis of MSE -- Bennett's integral for VQ
- High-resolution analysis of optimal performance -- Zador-Gersho formula
- Comparison to Shannon's rate-distortion theory analysis of optimal performance
- Product, polar, lattice, two-stage, tree-structured, hierarchical, ...

OPTIMALITY PROPERTIES (Useful in design and implementation)

Property 1: Given a codebook $C = \{\underline{w}_1,\ldots,\underline{w}_M\}$, let

$$V_i = \{\underline{x} : ||\underline{x}-\underline{w}_i|| < ||\underline{x}-\underline{w}_j||, \text{ for all } j \neq i\}.$$

A partition $S = \{S_1,\ldots,S_M\}$ minimizes MSE for the given codebook and random source vector $\underline{X}$ if and only if

$$S_i \doteq V_i \text{ for each } i, \qquad (*)$$

where $A \doteq B$ means $\Pr(\underline{X} \in (A-B) \cup (B-A)) = 0$.

Interpretation/Derivation:

The role of the encoder is to control the decoder to produce the best reproduction among all that it can produce. Therefore, if the source vector $\underline{x}$ is closer to $\underline{w}_i$ than to any other codevector, then to minimize MSE, $\underline{x}$ should be quantized to $\underline{w}_i$; i.e. $\underline{x}$ should be in S_i. This is Prop 1, except that Prop 1 recognizes that MSE is not affected if the S_i differs from V_i's by a set of probability zero.

Clearly, a partition satisfying $(*)$ has smallest possible MSE for the given codebook; i.e. it cannot be substantively improved. Conversely, if a partition does not satisfy $(*)$, it could be substantively improved, so it could not have smallest MSE.

Note: There are always points not in any V_i, namely points for which there are two or more closest codevectors. However, the set of such points has zero volume because it is a (k-1)-dimensional subset of R^k. Consequently, it has zero probability. Such points could be assigned to any cell, however, they would normally be assigned to the cell corresponding to one of the closest codevectors.

Notes:

- A partition S such that $S_i \supseteq V_i$ for all i is called a "Voronoi partition", and its cells are called "Voronoi cells". Other names for this partition are "nearest neighbor", "Dirichlet".

  Voronoi partitions are unique except for the points that are not contained in any of the V_i's. That is,

  $$S_i = V_i \cup T_i$$

  where T_i is some subset (possibly empty) of the points that are closest to $\underline{w}_i$ as well to some other point, that is of the subset

  $$\{\underline{x} : ||\underline{x}-\underline{w}_i|| = ||\underline{x}-\underline{w}_j|| \text{ some } j, \text{ and } ||\underline{x}-\underline{w}_i|| \le ||\underline{x}-\underline{w}_j|| \text{ for all } j\}$$

  All T_i's have probability zero.

- Ordinarily, we won't fuss about the sets of probability zero and the assignment of points that are equidistant between codevectors and simply say that "the optimal partition or the Voronoi partition is"

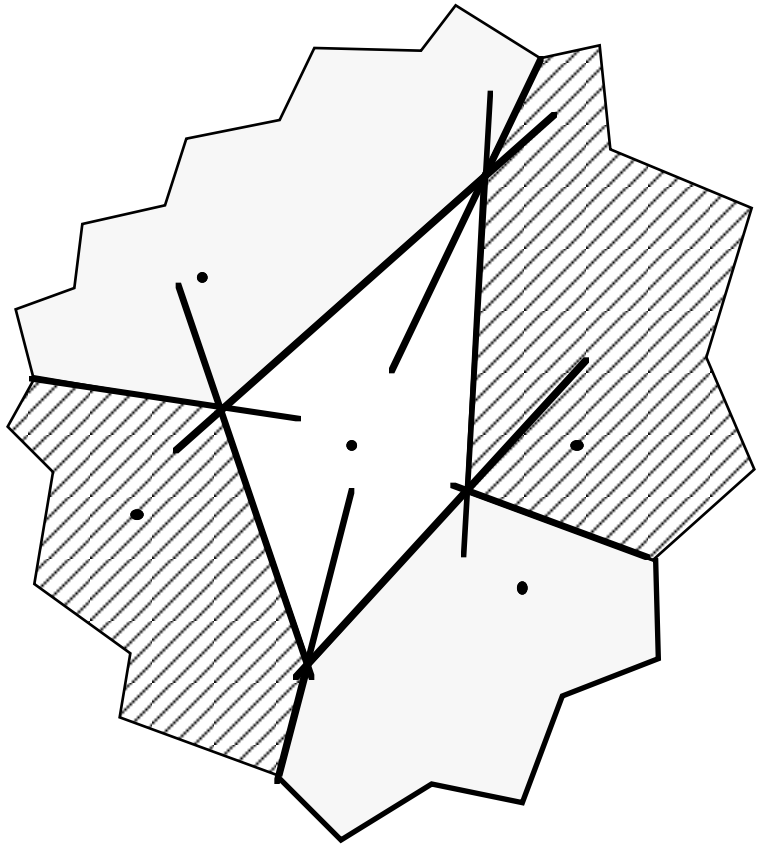
  $$S_i = \{\underline{x} : ||\underline{x}-\underline{w}_i|| < ||\underline{x}-\underline{w}_j||, \text{ for all } j \ne i\}$$

  or

  $$S_i = \{\underline{x} : ||\underline{x}-\underline{w}_i|| \le ||\underline{x}-\underline{w}_j||, \text{ for all } j \ne i\}.$$

- To find Voronoi partition, draw perpendicular bisectors between each pair of codevectors. These are (k-1)-dimen'l hyperplanes, each dividing R^k into two half spaces. The Voronoi region S_i is the intersection of the halfspaces containing $\underline{w}_i$.
- Voronoi cells are convex polyhedra.

- 2-Dimensional Example:



- Exercise: Show that if three points are not colinear, their three perpendicular bisectors meet a point.

Property 2: Given a partition $S = \{S_1, \ldots, S_M\}$ and source density $f_{\underline{X}}(\underline{x})$, the codevectors that minimize MSE are the "centroids"

$$\underline{w}_i = E[\underline{X}|\underline{X}\in S_i] = \int \underline{x}\, f_{\underline{X}}(\underline{x}|\underline{X}\in S_i)\, d\underline{x}\ ,\ i = 1,\ldots,M \qquad (**)$$

where

$$f_{\underline{X}}(\underline{x}|\underline{X}\in S_i) = \begin{cases} \dfrac{f_{\underline{X}}(\underline{x})}{\Pr(\underline{X}\in S_i)}, & \underline{x} \in S_i \\ 0\ , & else \end{cases}$$

Interpretation/Derivation:

The role of the decoder is to make the best estimate of the source output $\underline{X}$, given its input, which is, effectively, knowledge of the cell in which $\underline{X}$ lies.

When the decoder is told that $\underline{X}$ lies in cell S_i, its output, namely $\underline{w}_i$, should be the minimum MSE estimate of $\underline{X}$ given this knowledge, i.e. $\underline{w}_i = E[\underline{X}|\underline{X}\in S_i]$. Any other choice leads to larger MSE. (Recall that $E||\underline{X}-\underline{c}||^2$ is minimized by $\underline{c} = E\,\underline{X}$.)

If this is not already clear, it can be seen from the following:

$$D = \sum_{i=1}^{M} \Pr(\underline{X}\in S_i) \int_{S_i} (\underline{x}-\underline{w}_i)^2 \frac{f_{\underline{X}}(\underline{x})}{\Pr(\underline{X}\in S_i)}\, d\underline{x} = \sum_{i=1}^{M} \Pr(\underline{X}\in S_i) \int (\underline{x}-\underline{w}_i)^2 f_{\underline{X}}(\underline{x}|\underline{X}\in S_i)\, d\underline{x}$$

$$= \sum_{i=1}^{M} \Pr(\underline{X}\in S_i)\, E[(\underline{X}-\underline{w}_i)^2|\underline{X}\in S_i]$$

The sum is minimized by minimizing each term, i.e. by choosing $\underline{w}_i = E[\underline{X}|\underline{X}\in S_i]$.

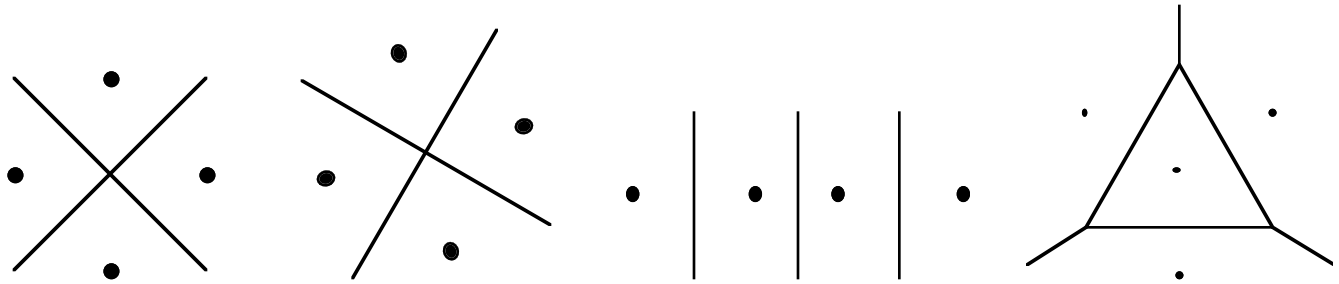
Note: The centroid of a convex cell is contained in the cell.

Corollary: Given a size M and a source density $f_{\underline{X}}(\underline{x})$, the best VQ (i.e. the one with smallest MSE) satisfies (*) and (**), which are called the "optimality properties" or "optimality criteria".

Notes:

- There may be more than one optimal quantizer.
- Even when there is only one optimal quantizer there may be more than one quantizer that satisfies the optimality criteria, in which case the best quantizer is one of them.

  Example: k =2, M=4, (X_1,X_2) is IID Gaussian. All of the following satisfy optimality criteria.



  Which is best?

Fact: A quantizer is locally optimal iff it satisfies (*) and (**)

Defn: A VQ is locally optimal if all sufficiently small perturbations increase or maintain distortion.

What is meant by "sufficiently small perturbation"?

Replace a codevector $\underline{w}_i$ by $\underline{w}_i + \varepsilon \underline{z}$ for an arbitrary vector $\underline{z}$ and some scalar ε. If the VQ is locally optimal, then for any $\underline{z}$ there is an ε_o such that for all $\varepsilon \le \varepsilon_o$, the perturbed VQ has the same or larger distortion as the original VQ. Any number of codevectors can be perturbed.

What about "sufficiently small perturbations" of the partition?

Imagine moving or stretching some boundary of some cell by an amount "proportional" to ε. Then there must exist some ε_o such that for all $\varepsilon \le \varepsilon_o$, the perturbed VQ has the same or larger distortion. A VQ is locally optimal if for all possible perturbations (of any number of codevectors and any number of ways of changing boundaries), there is an ε_o such that for all $\varepsilon \le \varepsilon_o$, the perturbed VQ has the same or large distortion.

Sketch of Proof of Fact: Local opt $\Rightarrow$ (*) and (**): If a quantizer does not satisfy (*) and (**), then it can be improved by small perturbations, so it is not a local optimal. The contrapositive of this is: Local opt $\Rightarrow$ (*) and (**).

(*) and (**) $\Rightarrow$ local opt: If a quantizer satisfies (*) and (**), then it is locally optimal because any small perturbation will cause it not to satisfy the (*) and (**), and in either case the MSE will increase.

BRUTE FORCE IMPLEMENTATION OF "UNSTRUCTURED" VQ

Full-Search Encoding

Store the codebook $C = \{\underline{w}_1,\ldots,\underline{w}_M\}$.

1. Given $\underline{x}$ compute $||\underline{x}-\underline{w}_i||^2$ for each i
2. Find the i that minimizes $||\underline{x}-\underline{w}_i||^2$ and send $\underline{c}_i$

This method encodes uses the Voronoi partition S for C.
C can be any codebook whatsoever.

Table-Lookup Decoding

Store the codebook $C = \{\underline{w}_1,\ldots,\underline{w}_M\}$.

When the decoder is given $\underline{Z} = \underline{c}_i$, it outputs $\underline{Y} = \underline{w}_i$ as the reproduction of $\underline{X}$.

This is the basic form of unstructured VQ. When people speak of ordinary VQ, this is often what they mean.

COMPLEXITY OF UNSTRUCTURED VQ

storage: codebook must be stored at encoder and decoder

	storage	ops/sample
encoder:	Mkb	3M
decoder:	Mkb	0

b = no. of bits/component $\cong$ R + 3 to 5 is usually sufficient,

encoding operations:

M distance squared's, each requiring k subtracts,
k squarings, (k-1) adds, M-1 comparisons

The "curse of dimensionality"

Since $M=2^{kR}$, both storage and computation increases exponentially with k and R.

The dimension-rate product kR is the key.

This is a big limitation. Generally,

$kR \leq 10$ or 12.

While modern computers can implement larger codebooks, it turns out that designing them can be extremely difficult.

There are some faster encoding algorithms, but not have been proven to have less than exponential complexity.

There are also:

- Fast search methods for unstructured VQ coebooks.

  We will discuss some later.

  Though they can be considerably simpler, I've yet to see one whose complexity does not increase exponentially with distortion-rate product.

- Structured VQ methods

  e.g. JPEG

  We will discuss other later.

  Their codebooks and/or partitions are structured in a way that permits fast encoding.

  Their partition might not be Voronoi.

  Their codevectors might not be centroids.

  Their performance might not be as good as an "optimal" VQ with the same dimension and rate, but their lower complexity might permit a larger dimension and better performance for the same complexity.

  (LOOK BACK TO OUR EXAMPLES)

GENERALIZED LLOYD VQ DESIGN ALGORITHMS

These are iterative algorithms, in the spirit of Lloyd's algorithm for scalar quantizer design. They seek a locally optimum quantizer by alternating between finding the best partition for the most recently found codebook and finding the best codebook for the most recently found partition. They stop when (*) and (**) are nearly satisfied; i.e. when a locally optimum VQ is approached. There are two basic types:

A. Design from pdf

B. Design from training sequence

A. Design from pdf.

Stop when the centroids and/or the distortion changes negligibly.

Convergence of the algorithm:

Since each step (finding Voronoi partition or centroids) does not increase distortion, the distortion of the algorithm is guaranteed to converge.

Typically, the actual partition and codebook converge to a local optimum (i.e. a VQ that satisfies (*) and (**)), which might or might not be the globallyt optimal quantizer. But it is conceivable that it might get into a cycle where, for example, it alternates between two different quantizers.

Initial codebook:

There are many possible choices. Some common one:

(a) A set of M representative source sequences. They might be generated from the pdf with a random number generator.

(b) The k-fold Cartesian product of an optimal scalar quantizer with $M^{1/k}$ levels.

(c) The set formed by adding an additional codevector in close proximity to each codevector of an optimal k-dimensional VQ with M/2 codevectors. In this method, one starts by designing a VQ with 2 codevectors, then successively designs VQ's with twice the size of the previous one.

The "design from pdf" Lloyd algorithm is seldom used

It is often used for designing scalar quantizers.

It is seldom used for designing vector quantizers (k>1) because

- the pdf of the source vector is often unknown.
- working with a k-dimensional pdf's (e.g. computing the k-dimensional integrals involved in centroid calculations) is prohibitively complex.

Therefore, in practice VQ's are almost always designed by "training sequence" methods such as that described next.

B. Design from training sequence. (LBG algorithm)

Given a training sequence $\underline{t}_1, \underline{t}_2, \ldots, \underline{t}_N$, i.e. a representative sequence of k-dimensional vectors from the source.

Choose an initial codebook $C = \{\underline{w}_1, \ldots, \underline{w}_M\}$.

Iterate the following steps until the centroids change little and/or the distortion changes little:

1. Find Voronoi "sets": $\tilde{S}_i = \{ \underline{t}_i : \underline{t}_i$ closer to $\underline{w}_i$ than to any other codevector$\}$
2. Find empirical centroids: $\tilde{\underline{w}}_i = \frac{1}{N_i} \sum_{j : \underline{t}_j \in \tilde{S}_i} \underline{t}_j$, where N_i = # training vectors in $\tilde{S}_i$.

Step 1 is implemented by constructing a table in which each training sequence vector is labeled with the index of the codevector to which it is closest.

Step 2 is implemented by counting and averaging all training vectors with a given label.

Alternately, iterate the following single step: given $\underline{w}_1, \ldots, \underline{w}_M$, find new centroids via

$$\tilde{\underline{w}}_i = \frac{1}{N_i} \sum_{j : \underline{t}_j \text{ closest to } \underline{w}_i} \underline{t}_j, \text{ where } N_i = \text{\# train'g vect's closest to } \underline{w}_i$$

Here, for each training vector $\underline{t}_j$,

- find codevector, say $\underline{w}_i$, to which it is closest, via full search.
- increment a counter that stores N_i
- add $\underline{t}_j$ to an accumator that computes the sum in $\tilde{\underline{w}}_i$.

Initial codebook:

The same choices as with Algorithm A are available.

We have a version of this algorithm available, written in C.

LBG = Linde, Buzo and Gray, who published it in IEEE Trans. Commun., Jan. 1980

Convergence of the algorithm:

Since each step of the algorithm maintains or decreases the training distortion, i.e. the distortion measured on the training sequence itself, the training distortion will necessarily converge.

Typically, the partition and codebook generated by the algorithm are converging to a local optimum, i.e. to a pair that satisfy the empirical versions of (*) and (**)).

Since there are only a finite number of distinct partitions of the given training sequence, after a finite number of steps the algorithm must necessarily reach a local optimum after which it does not change, or it must cycle repeatedly through some finite number of paritition-codebook pairs.

However, the algorithm is usually stopped long before either of these occur.

Because the algorithm deals only with a finite set of training vectors, there tend to be more local minimum than with algorithms based that work directly with the pdf such as Algorithm A. Therefore, it is usually wise to rerun the algorithm with several different choices of initial codebook.

Training distortion vs. actual distortion:

The *actual distortion* $\left(D = E(\underline{X}-Q(\underline{X}))^2 \text{ for the given source}\right)$ of the quantizer designed by this algorithm is greater than the *training distortion*, which is the distortion measured on the training sequence.

To see why, consider the extreme case where the training sequence length N equals the size M of the desired VQ. In this case, the algorithm will choose the codebook to be the training sequence itself, and it will find the training distortion to be zero, whereas the actual distortion will ordinarily be far from the mimimum possible distortion.

The difference between training and actual distortion becomes smaller as N increases, but is nevertheless usually significant.

Because training distortion can be substantially smaller than actual distortion, it is customary to estimate the actual distortion by running the VQ on a *test sequence* that is distinct from the training sequence.

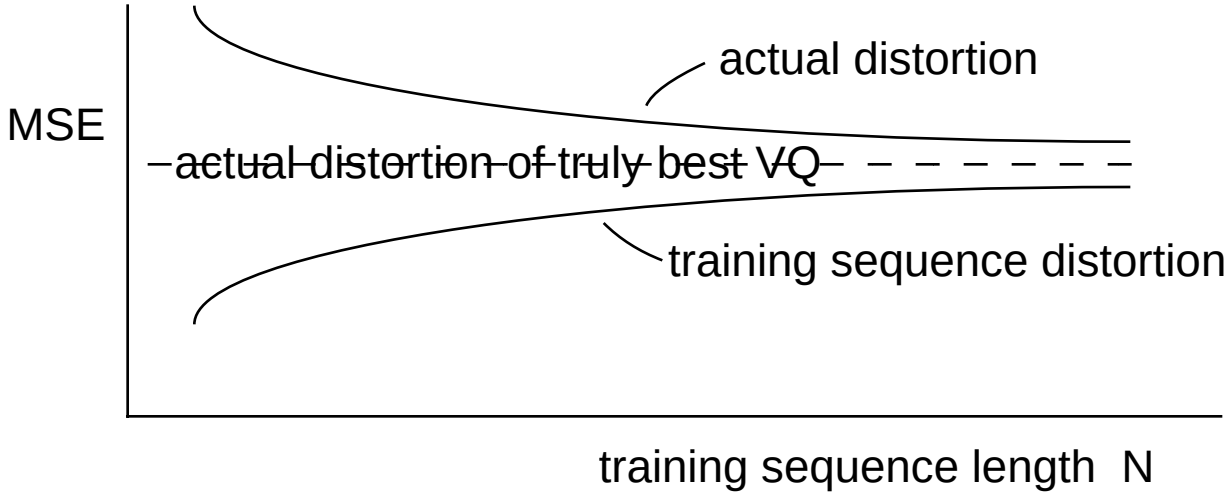
Training sequence length:

How large to make the training sequence length N?

Two issues:

- the VQ designed by the algorithm becomes better as N is made larger,
- the training distortion becomes a more accurate estimate of actual distortion.

The convergence of the actual and training distortions with increasing N are illustrated in the figure below. A typical rule of thumb is that to design a good VQ, N should be at least 50 M, and larger is better. Significantly larger N is needed if training distortion is to be used as an estimate of actual distortion.



A conservative strategy is to choose N large enough that the training and test sequence distortions are reasonably close.

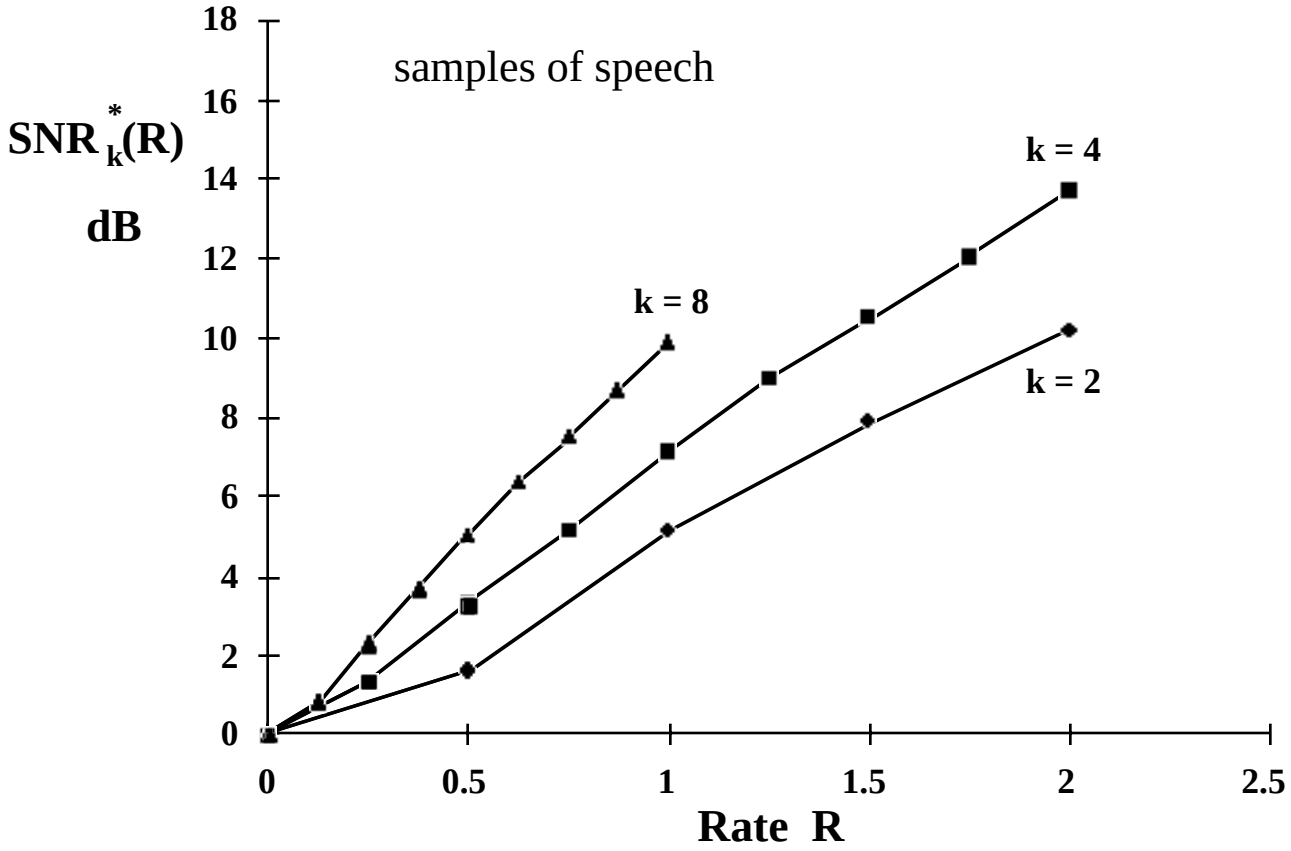
Complexity of VQ design:

The above design algorithm involves encoding the training sequence a number of times and, additionally, computing the new centroids. Hence, the number of operations performed is approximately proportional to NM times the number of iterations.

Since a VQ is ordinarily designed once and used many, many times, we are willing to live with very complex VQ design algorithms. Nevertheless, design becomes a genuine problem for VQ's with moderate to large dimension-rate products. (I can't recall having seen an ustructured VQ designed with $kR > 14$). Thus, there are dimension-rate products e.g. $kR = 15$ for which one could conceivably implement a VQ, but for which design is infeasible.

Not surprisingly a number of reduced complexity design algorithms have been proposed. These typically involve a fast encoding algorithm such as those described previously. Another way to speed the algorithm is to make a good choice of the initial codebook. This reduces the number of iterations required.

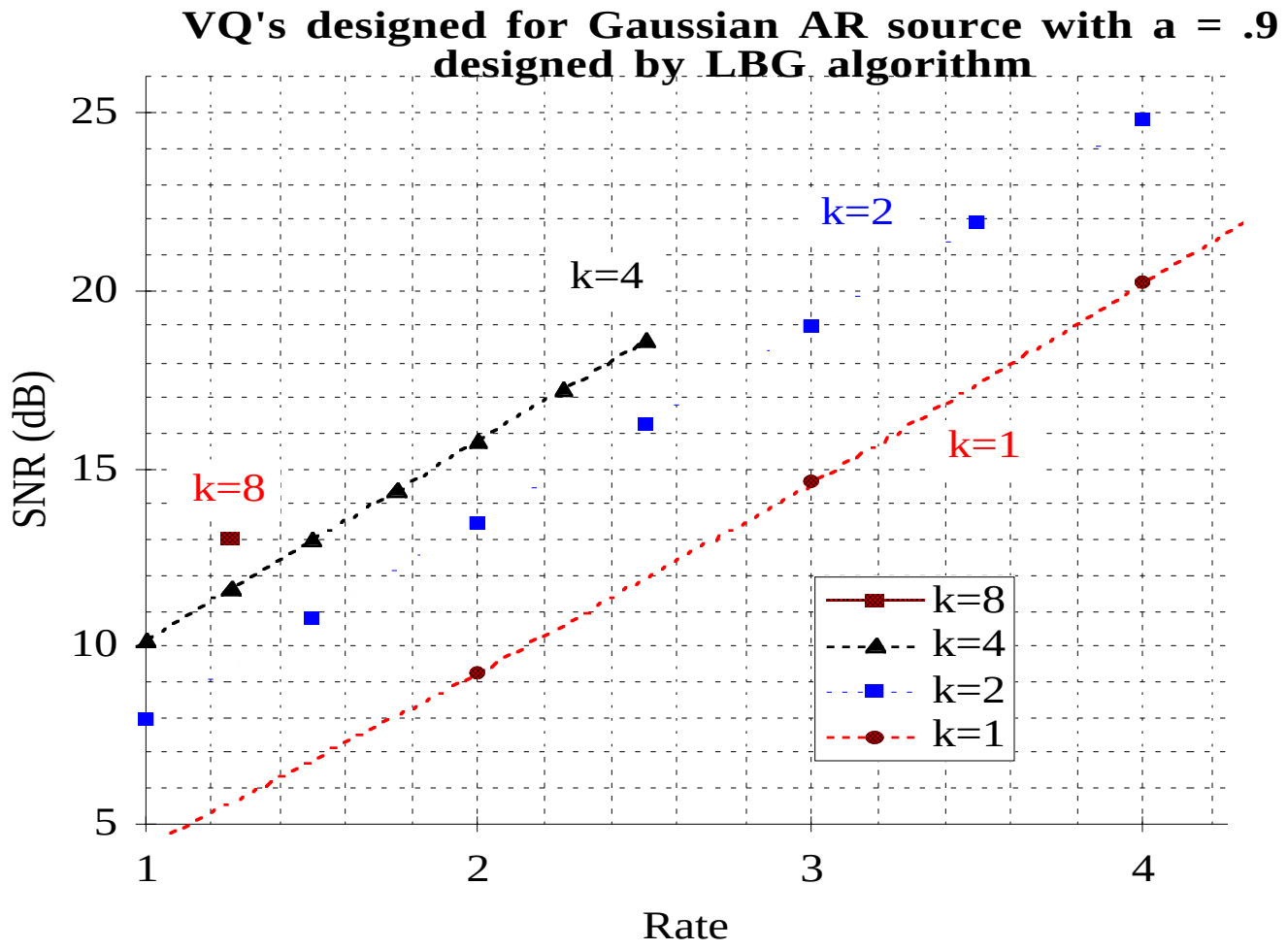
EXAMPLES OF VQ'S DESIGNED BY THE LBG ALGORITHM



k-dimensional VQ's designed using LBG algorithm on training sequence of 640,000 samples of speech taken at 6.5 kHz sampling rate.

Notice the gains with increasing dimension.

Notice also that for k=8, the quantizers were designed only to rate 1. This is because of the large complexity of designing higher rate VQ's with dimension 8.

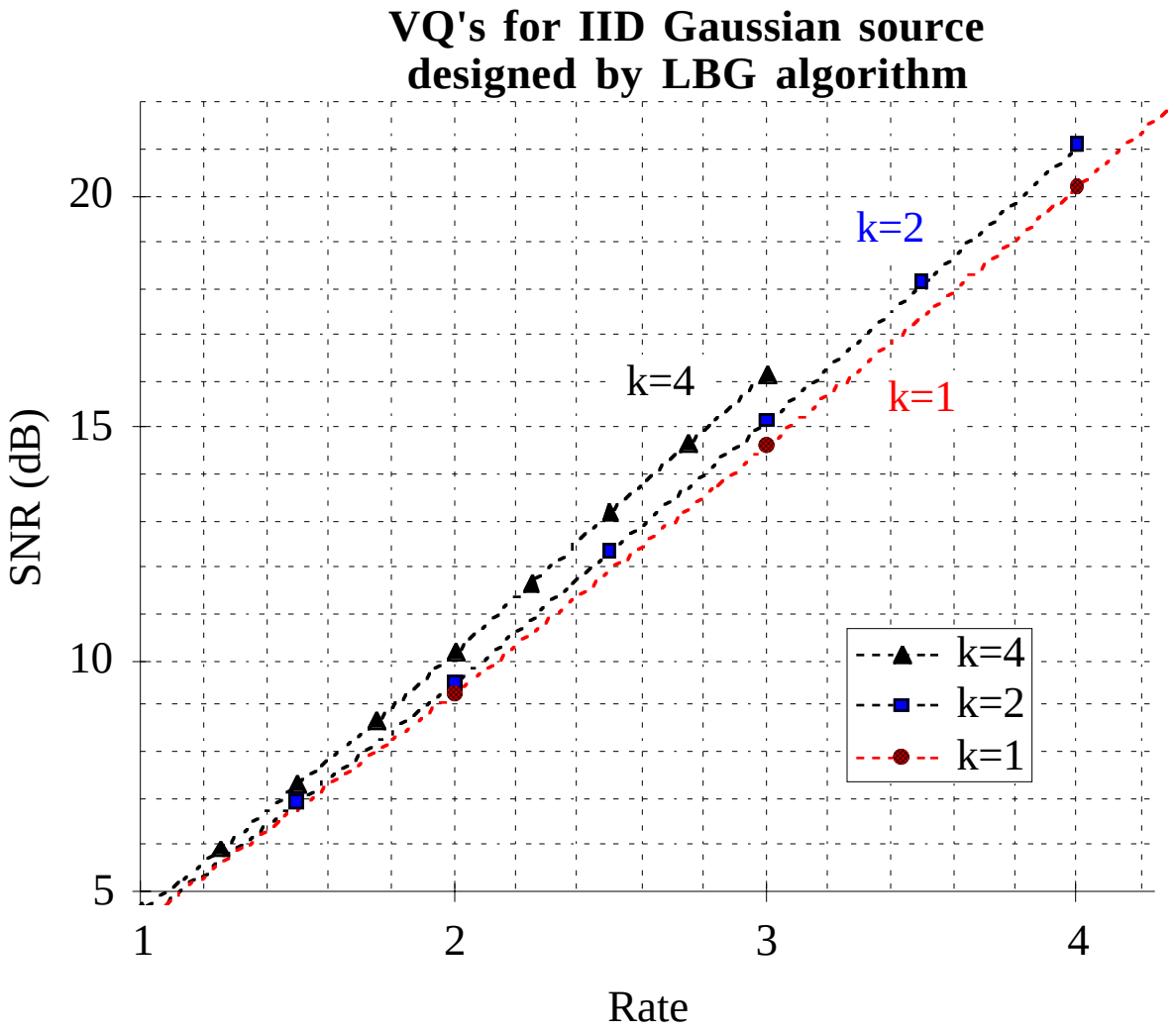


Gains of VQ over scalar Q:

k	=	2	4	8	
gain	≅	4.2	6.5	7.8	dB

Important questions:

Why does VQ gain over scalar Q? Why does gain increase with dimension?



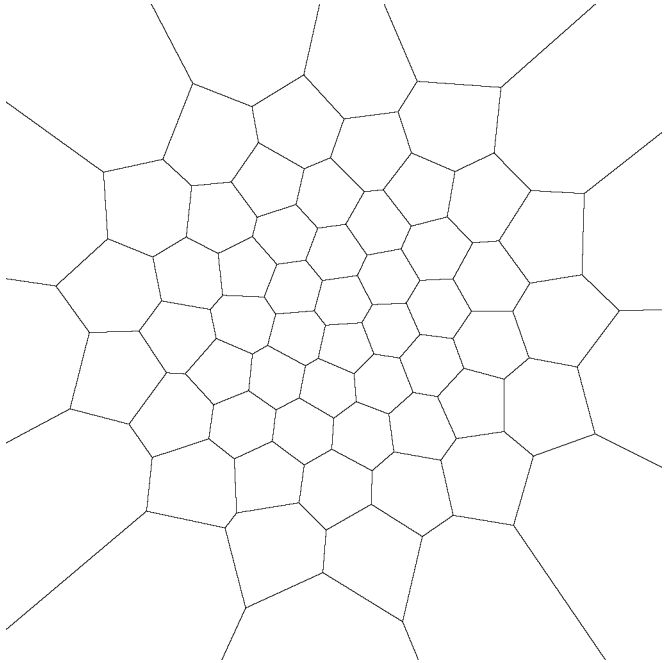
Again notice the gains of VQ over scalar quant (at rate 3, k=4 VQ gains 1.6 dB).

Important questions:

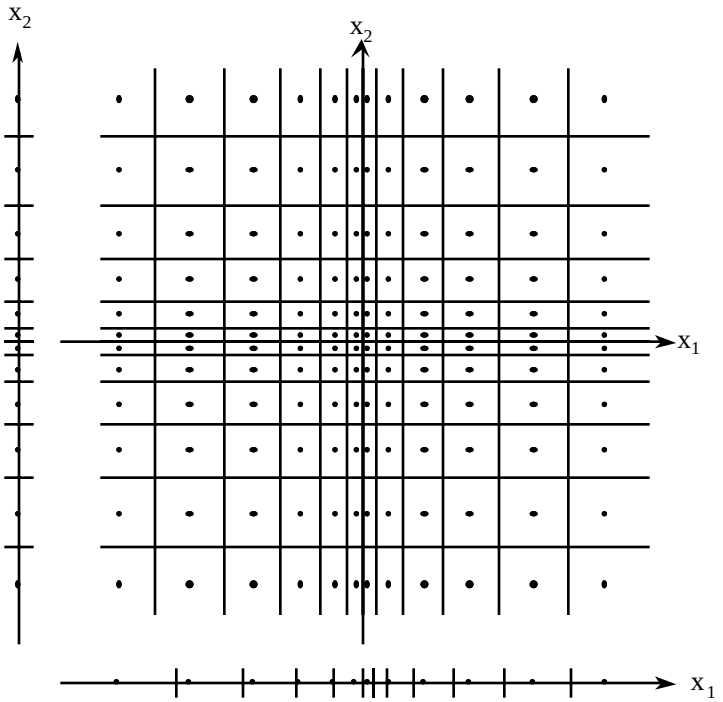
Why does VQ gain over scalar Q? Why does gain increase with dimension?

COMPARISON OF QUANTIZERS FOR AN IID GAUSSIAN SOURCE

An optimal 2-dimensional VQ



An optimal scalar quantizer used twice



How are the partitions and codebooks different?

HISTORICAL NOTE

It wasn't until the late 1970's that the VQ was seriously proposed and the LBG algorithm was developed. Before that, information theorists generally assumed that large dimensions would be needed for VQ's to produce significant gains over scalar quantization (probably due to their experience with channel codes), and because they could easily see the large complexity associated with even moderately large dimensions.

The LBG algorithm is also used in pattern recognition for identifying clusters of multidimensional features. It was developed independently in that community as the "k-means algorithm".

OTHER PROPERTIES OF OPTIMAL QUANTIZERS

Consider a VQ with

$$C = \{\underline{w}_1,\ldots,\underline{w}_M\}, \quad S = \{S_1,\ldots,S_M\}, \quad P_i = \Pr(\underline{X}\in S_i), \quad \underline{Y} = Q(\underline{X}) = \text{column vector.}$$

Fact: If C satisfies the centroid property ($\underline{w}_i = E[\underline{X}|\underline{X}\in S_i]$) for all i, then

1. $E\,\underline{Y} = E\,\underline{X}$
2. $E\,X_m Y_n = E\,Y_m Y_n$
3. $E\,\underline{X}^t\underline{Y} = E\,||\underline{Y}||^2$
4. $E\,Y_m(X_n - Y_n) = 0$
5. $E\,\underline{Y}^t(\underline{X}-\underline{Y}) = 0$
6. $E\,||\underline{Y}||^2 = E\,||\underline{Y}||^2 - E\,||\underline{X}-\underline{Y}||^2$

Proofs: Assume C satisfies centroid property ($\underline{w}_i = E[\underline{X}|\underline{X} \in S_i]$).

1. $E\,\underline{Y} = E\,\underline{X}$

   Pf: $E\,\underline{Y} = \sum_{i=1}^{M} P_i\, E[\underline{Y}|\underline{X} \in S_i] = \sum_{i=1}^{M} P_i\, \underline{w}_i = \sum_{i=1}^{M} P_i\, E[\underline{X}|\underline{X} \in S_i] = E\,\underline{X}$

2. $E\,X_m Y_n = E\,Y_m Y_n$

   Pf: $E\,X_m Y_n = \sum_{i=1}^{k} \Pr(\underline{X} \in S_i)\, E[X_m Y_n|\underline{X} \in S_i]$

   $= \sum_{i=1}^{k} \Pr(\underline{X} \in S_i)\, E[X_m w_{i,n}|\underline{X} \in S_i] = \sum_{i=1}^{k} \Pr(\underline{X} \in S_i)\, w_{i,n}\, w_{i,m}$

   $E\,Y_m Y_n = \sum_{i=1}^{k} \Pr(\underline{X} \in S_i)\, E[Y_m Y_n|\underline{X} \in S_i]$

   $= \sum_{i=1}^{k} \Pr(\underline{X} \in S_i)\, w_{i,n}\, w_{i,m} = E\,X_m Y_n$

3. $E\,\underline{X}^t\underline{Y} = E\,||\underline{Y}||^2$

   Pf: $E\,\underline{X}^t\underline{Y} = \sum_{i=1}^{k} E\,X_i Y_i = \sum_{i=1}^{k} E\,Y_i Y_i = E \sum_{i=1}^{k} Y_i^2 = E\,||\underline{Y}||^2$

   (by 2)

4. $E\,Y_m(X_n - Y_n) = 0$

   Pf: $E\,Y_m(X_n - Y_n) = E\,Y_m X_n - E\,Y_m Y_n = E\,Y_m Y_n - E\,Y_m Y_n = 0$

   (by 2)

5. $E\,\underline{Y}^t(\underline{X}-\underline{Y}) = 0$

Pf: $E\,\underline{Y}^t(\underline{X}-\underline{Y}) = E\,\underline{Y}^t\underline{X} - E\,\underline{Y}^t\underline{Y} = E\,||\underline{Y}||^2 - E\,||\underline{Y}||^2 = 0$

(by 3)

6. $E\,||\underline{Y}||^2 = E\,||\underline{Y}||^2 - E\,||\underline{X}-\underline{Y}||^2$

Pf: $E\,||\underline{X}-\underline{Y}||^2 = E\,||\underline{X}||^2 - 2\,E\,\underline{X}^t\underline{Y} + E\,||\underline{Y}||^2$

$= E\,||\underline{X}||^2 - 2\,E\,||\underline{Y}||^2 + E\,||\underline{Y}||^2 = E\,||\underline{X}||^2 - E\,||\underline{Y}||^2$

(by 3)