**Addendum to DPCM and Transform Coding Lectures:**

**Fact:** If $X$ is a wide-sense stationary, zero-mean random process, then

$$
\left| K^{(n+1)} \right| = M_n \left| K^{(n)} \right|
$$

where $M_n$ denotes the mean-squared error of the best linear predictor for $X_i$ from $X_{i-n}, \ldots, X_{i-n}$, and $K^{(n)}$ and $K^{(n+1)}$ denote, respectively, the covariance matrices of $X_1, \ldots, X_n$ and $X_1, \ldots, X_{n+1}$, and $\left| K^{(n+1)} \right|$ and $\left| K^{(n)} \right|$ denote their determinants.

**Proof:** We'll use Cramer's rule for solving systems of linear equations and some standard facts regarding determinants. Let $r_i = E[X_nX_{n+i}]$ and $\mathbf{r} = [r_1, r_2, \ldots, r_n]^{t}$. Then in terms of the $r_i$'s, we showed in lecture that

$$
M_n = r_0 - r^{t} (K^{(n)})^{-1} \mathbf{r}.
$$

(Note that $r_0 = \text{var}(X_i)$.)

Moreover,

$$
K^{(n+1)} = \begin{bmatrix}
    r_0 & r_1 & \cdots & r_{n-1} & r_n \\
    r_1 & r_0 & \cdots & r_{n-2} & r_{n-1} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    r_{n-1} & r_{n-2} & \cdots & r_0 & r_1 \\
    r_n & r_{n-1} & \cdots & r_1 & r_0
\end{bmatrix}
\quad \text{and} \quad
K^{(n)} = \begin{bmatrix}
    r_0 & r_1 & \cdots & r_{n-1} \\
    r_1 & r_0 & \cdots & r_{n-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    r_{n-1} & r_{n-2} & \cdots & r_0
\end{bmatrix}.
$$

For future reference, observe that removing the first row and column of $K^{(n+1)}$ leaves $K^{(n)}$. Let's begin with the expansion of the determinant of $K^{(n+1)}$ along its first row:

$$
\left| K^{(n+1)} \right| = r_0 \left| M_1 \right| - r_1 \left| M_2 \right| + r_2 \left| M_3 \right| + \ldots + (-1)^n r_n \left| M_{n+1} \right|
$$

where $M_i$ is $K^{(n+1)}$ with the first row and ith column removed and where it is easy to see that $\left| M_i \right| = \left| K^{(n)} \right|$. ($M_i$ is called a minor of $K^{(n+1)}$.)

Next, let $\mathbf{s} = [s_1, s_2, \ldots, s_n] = (K^{(n)})^{-1} r$. By Cramer's rule, $s_i = \frac{1}{\left| K^{(n)} \right|} \left| B_i \right|$, where $B_i$ is $K^{(n)}$ with its ith column replaced by $r$. Therefore

$$
r^{t} (K^{(n)})^{-1} \mathbf{r} = r^{t} \mathbf{s} = \frac{1}{\left| K^{(n)} \right|} \sum_{i=1}^{n} r_i \left| B_i \right|.
$$

Therefore,

$$
M_n \left| K^{(n)} \right| = \left( r_0 - r^{t} (K^{(n)})^{-1} \mathbf{r} \right) \left| K^{(n)} \right| = r_0 \left| K^{(n)} \right| - \sum_{i=1}^{n} r_i \left| B_i \right|.
$$
Comparing the expressions for $|K^{(n+1)}|$ and $M_n|K^{(n)}|$, we see that it remains only to show that

$$-|B_i| = |M_{i+1}|(-1)^i, \quad i = 1, \ldots, n.$$ 

To do so, observe that

$$M_{i+1} = \begin{bmatrix} \tilde{R}^{(n)}_i \end{bmatrix}, \quad i = 1, \ldots, n,$$

where $\tilde{R}^{(n)}_i$ is $R^{(n)}$ with its ith column removed. We now see that moving the first column of $M_i$ to the right (i-1) places creates the matrix $B_i$. Since moving a column to the right one place multiplies the determinant by -1, we have

$$|B_i| = |M_{i+1}|(-1)^{i-1} = -|M_{i+1}|(-1)^i, \quad i = 1, \ldots, n,$$

which is just what we needed to show, and which completes the proof.