Addendum to DPCM and Transform Coding Lectures:

Fact: If X is a wide-sense stationary, zero-mean random process, then

$$\left|K^{(n+1)}\right| = M_n \left|K^{(n)}\right|$$

where M_n denotes the mean-squared error of the best linear predictor for X_i from $X_{i-n},\ldots,X_{i-n}$, and $K^{(n)}$ and $K^{(n+1)}$ denote, respectively, the covariance matrices of $X_1,\ldots,X_n$ and $X_1,\ldots,X_{n+1}$, and $\left|K^{(n+1)}\right|$ and $\left|K^{(n)}\right|$ denote their determinants.

Proof: We'll use Cramer's rule for solving systems of linear equations and some standard facts regarding determinants. Let $r_i = E[X_n X_{n+i}]$ and $\underline{r} = [r_1,r_2,\ldots,r_n]^t$. Then in terms of the r_i's, we showed in lecture that

$$M_n = r_0 - \underline{r}^t (K^{(n)})^{-1}\underline{r} \,.$$

(Note that $r_0 = \mathrm{var}(X)$.) Moreover,

$$K^{(n+1)} = \begin{bmatrix} r_0 & r_1 & . & . & . & r_{n-1} & r_n \\ r_1 & r_0 & . & . & . & r_{n-2} & r_{n-1} \\ & & . & & & & \\ r_{n-1} & r_{n-2} & . & . & . & r_0 & r_1 \\ r_n & r_{n-1} & . & . & . & r_1 & r_0 \end{bmatrix} \quad \text{and} \quad K^{(n)} = \begin{bmatrix} r_0 & r_1 & . & . & . & r_{n-1} \\ r_1 & r_0 & . & . & . & r_{n-1} \\ & & & . & & \\ r_{n-1} & r_{n-2} & . & . & . & r_0 \end{bmatrix}.$$

For future reference, observe that removing the first row and column of $K^{(n+1)}$ leaves $K^{(n)}$. Let's begin with the expansion of the determinant of $K^{(n+1)}$ along its first row:

$$\left|K^{(n+1)}\right| = r_0 \left|M_1\right| - r_1 \left|M_2\right| + r_2 \left|M_3\right| \ldots + (-1)^n r_n \left|M_{n+1}\right|$$

$$= r_0 \left|K^{(n)}\right| + \sum_{i=1}^{n} r_i \left|M_{i+1}\right| (-1)^i$$

where M_i is $K^{(n+1)}$ with the first row and ith column removed and where it is easy to see that $\left|M_1\right| = \left|K^{(n)}\right|$. ($M_i$ is called a minor of $K^{(n+1)}$.)

Next, let $\underline{s} = [s_1,s_2,\ldots,s_n]^t = (K^{(n)})^{-1}\underline{r}$. By Cramer's rule, $s_i = \dfrac{\left|B_i\right|}{\left|K^{(n)}\right|}$, where B_i is $K^{(n)}$ with its ith column replaced by $\underline{r}$. Therefore

$$\underline{r}^t (K^{(n)})^{-1}\underline{r} = \underline{r}^t \underline{s} = \frac{1}{\left|K^{(n)}\right|} \sum_{i=1}^{n} r_i \left|B_i\right|.$$

Therefore,

$$M_n \left|K^{(n)}\right| = \left(r_0 - \underline{r}^t (K^{(n)})^{-1}\underline{r}\right) \left|K^{(n)}\right| = r_0 \left|K^{(n)}\right| - \sum_{i=1}^{n} r_i \left|B_i\right| .$$

Comparing the expressions for $\left|K^{(n+1)}\right|$ and $\mathcal{M}_n \left|K^{(n)}\right|$, we see that it remains only to show that

$$-\left|B_i\right| = \left|M_{i+1}\right|(-1)^i, \quad i = 1,\ldots,n \ .$$

To do so, observe that

$$M_{i+1} = \left[\underline{r} \ \ \tilde{R}_i^{(n)}\right], \quad i = 1,\ldots,n,$$

where $\tilde{R}_i^{(n)}$ is $R^{(n)}$ with its ith column removed. We now see that moving the first column of M_i to the right (i-1) places creates the matrix B_i. Since moving a column to the right one place multiplies the determinant by -1, we have

$$\left|B_i\right| = \left|M_{i+1}\right|(-1)^{i-1} = -\left|M_{i+1}\right|(-1)^i, \quad i=1,\ldots,n,$$

which is just what we needed to show, and which completes the proof.