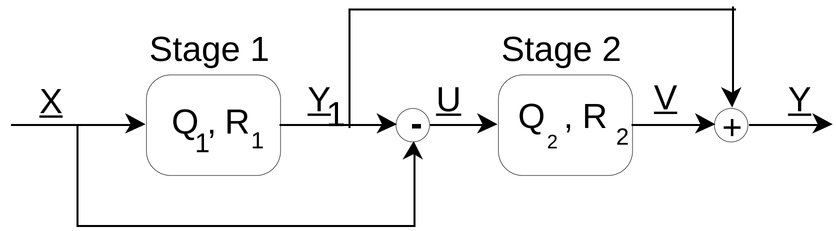
Two-Stage Vector Quantization (2VQ)

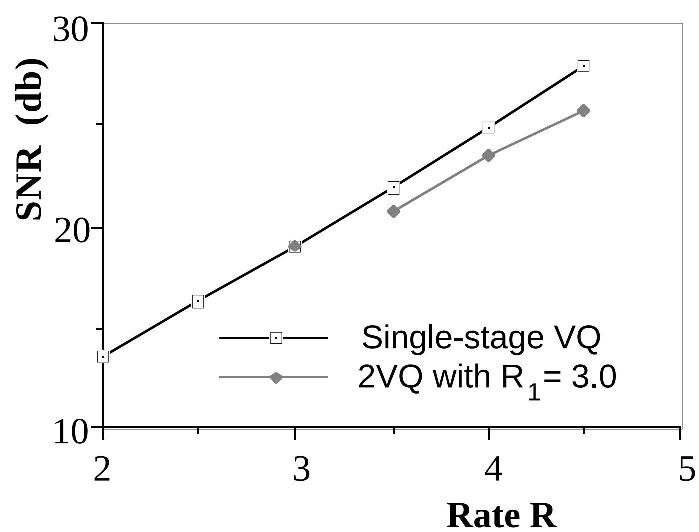


- Operation:
  1. Quantize $\underline{X}$ using k-dim'l vq Q_1 with rate R_1
  2. Quantize the error/residual $\underline{U} = \underline{X} - \underline{Y}_1$ using k-dim'l Q_2 with rate R_2
- Rate: $R = R_1 + R_2$
- Distortion: $D = \frac{1}{k} E\,||\underline{U} - \underline{V}||^2 =$ Stage 2 distortion, because $(\underline{X} - \underline{Y}) = (\underline{U} - \underline{V})$
- Complexity: much lower than single-stage VQ -- $M_1 + M_2$ vs. $M_1 M_2$
- Greedy Design:

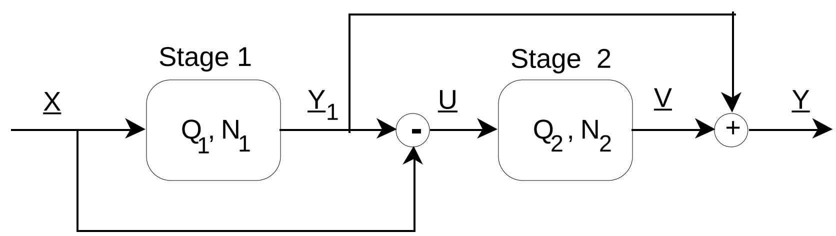
1. design Q_1 to minimize $E||\underline{X}-\underline{Y}_1||^2$,
2. design Q_2 to minimize $E||\underline{U}-\underline{V}||^2$, e.g. on train'g sequence of first-stage errors.
3. How to choose R_1 and R_2? Complexity minimized by choosing $R_1=R_2=R/2$.

Performance Example

Gaussian AR source, $\rho = .9$, VQ dimension $k = 2$



ANALYSIS OF 2VQ

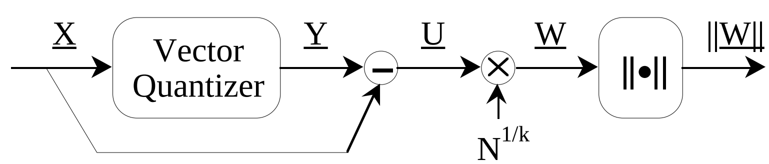


- Apply Bennett's integral to Stage 2

$$D = D_2 \cong \frac{1}{N_2^{2/k}} \int \frac{m_2(\underline{u})}{\lambda_2(\underline{u})^{2/k}} \, p_{\underline{U}}(\underline{u}) \, d\underline{u}$$

- Need to find the probability density $p_{\underline{U}}(\underline{u})$ of $\underline{U}$.

ASYMPTOTIC ERROR LENGTH DENSITY

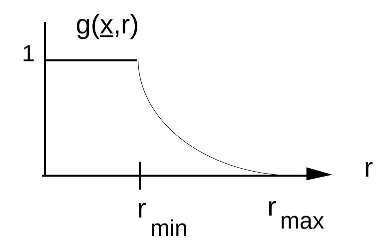


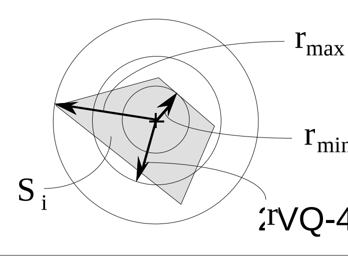
- Normalized Error: $\underline{W} = N^{1/k}\,\underline{U} = N^{1/k}(\underline{X}-\underline{Y})$
- For k-dimensional VQ with large N, mostly small cells, neighboring cells having similar sizes & shapes, point density $\lambda(\underline{x})$, and <u>shape profile</u> $g(\underline{x},r)$, the probability density of the normalized length of the quantization error is

$$p_{||\underline{W}||}(w) \cong k\, V_k\, w^{k-1} \int p_{\underline{X}}(\underline{x})\, \lambda(\underline{x})\, g(\underline{x}, w\, \lambda(\underline{x})^{1/k})\, d\underline{x},$$
$$w \geq 0,$$

where shape profile is

$g(\underline{x},r)$ = fraction of surface of sphere of radius r covered by cells S_i in the vicinity of $\underline{x}$.

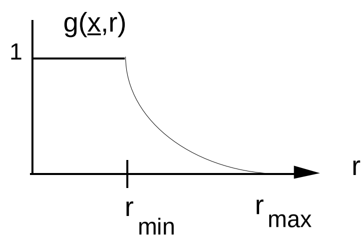


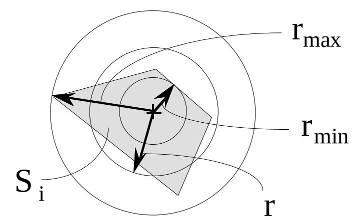


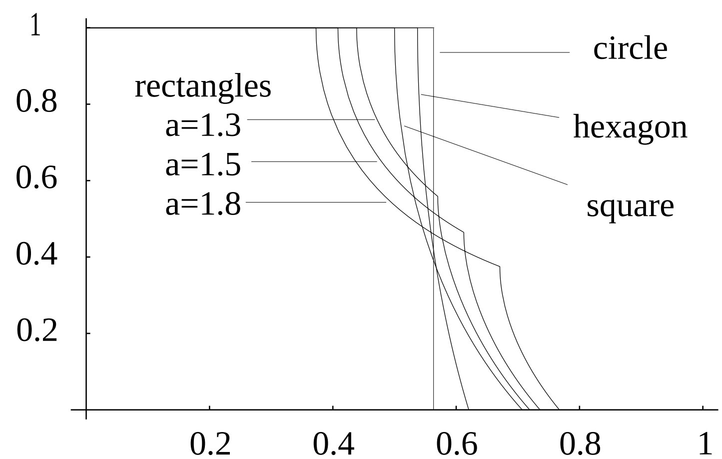
Cell & sphere centered at $\underline{y}_i$. Cell normalized to unit volume.

SHAPE PROFILE

$g(\underline{x},r)$ = fraction of surface of sphere of radius r covered by cells S_i in the vicinity of $\underline{x}$. (Cell & sphere centered at $\underline{y}_i$. Cell normalized to unit volume.)







2VQ-5

DERIVATION OF ASYMPTOTIC ERROR LENGTH DENSITY

For $w \geq 0$,

$$p_{||\underline{W}||}(w) = \frac{d}{dw} F_{||\underline{W}||}(w)$$

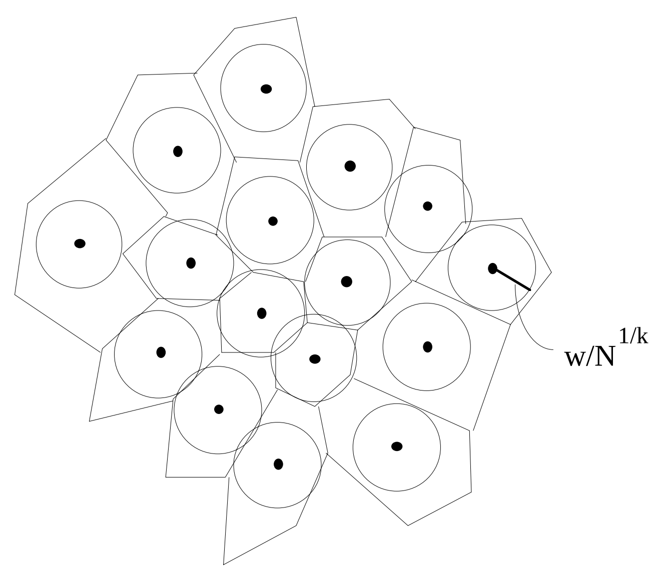
$$F_{||\underline{W}||}(w) = \Pr(\,||\underline{W}|| \leq w\,)$$

$$= \Pr\left(||\underline{X}-Q(\underline{X})|| \leq \frac{w}{N_{1/k}}\right)$$

$$= \Pr\left(\,\underline{X} \in \Sigma_{Q(\underline{X})}(wN^{-1/k})\right)$$

$$= \sum_{i=1}^{N} \Pr(\,\underline{X} \in S_i \cap \Sigma_{\underline{y}_i}(wN^{-1/k}))$$

$$= \sum_{i=1}^{N} \int_{S_i \cap \Sigma_{\underline{y}_i}(wN^{-1/k})} p_{\underline{X}}(\underline{x})\, d\underline{x}$$



2VQ-6

For $w \ge 0$,

$$F_{\|\underline{W}\|}(w) = \Pr(\|\underline{W}\| \le w) = \Pr\left(\|\underline{X}-Q(\underline{X})\| \le \frac{w}{N^{1/k}}\right) = \Pr\left(\underline{X} \in \Sigma_{Q(\underline{X})}(wN^{-1/k})\right)$$

$$= \sum_{i=1}^{N} \Pr(\underline{X} \in S_i \cap \Sigma_{\underline{y}_i}(wN^{-1/k})) = \sum_{i=1}^{N} \int_{S_i \cap \Sigma_{\underline{y}_i}(wN^{-1/k})} p_{\underline{X}}(\underline{x})\, d\underline{x}$$

$$\cong \sum_{i=1}^{N} p_{\underline{X}}(\underline{y}_i)\ \mathrm{vol}\left(S_i \cap \Sigma_{\underline{y}_i}(wN^{-1/k})\right) \quad \text{(small cells)}$$

$$= \sum_{i=1}^{N} p_{\underline{X}}(\underline{y}_i) \int_0^{wN^{-1/k}} \text{surf-area}\left(S_i \cap \Sigma_{\underline{y}_i}(r)\right) dr$$

$$= \sum_{i=1}^{N} p_{\underline{X}}(\underline{y}_i) \int_0^{wN^{-1/k}} g\left(\underline{y}_i, r\, \mathrm{vol}(S_i)^{-1/k}\right) k V_k r^{k-1}\, dr$$

$$= \sum_{i=1}^{N} p_{\underline{X}}(\underline{y}_i) \int_0^{w} g\left(\underline{y}_i, u\, \lambda(\underline{y}_i)^{1/k}\right) k V_k u^{k-1} du\ \ \lambda(\underline{y}_i)\, \mathrm{vol}(S_i)$$

$$\cong k V_k \int p_{\underline{X}}(\underline{x})\, \lambda(\underline{x}) \int_0^{w} g\left(\underline{x}, u\, \lambda(\underline{x})^{1/k}\right) u^{k-1}\, du\, d\underline{x}$$

Taking the derivative yields

$$p_{\|\underline{W}\|}(w) \cong k V_k w^{k-1} \int p_{\underline{X}}(\underline{x})\, \lambda(\underline{x})\, g\left(\underline{x}, w\lambda(\underline{x})^{1/k}\right) d\underline{x}, \quad w \ge 0$$

EXAMPLES

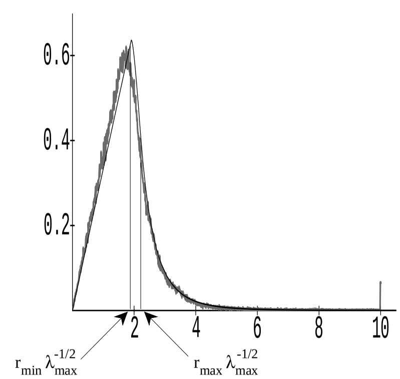
Histograms of normalized error length
Predictions based on formula for asymptotic error density

1. Memoryless Gaussian source

   Optimal quantizer designed by LBG algorithm

   k = 2, M = 256, R = 4

   Prediction based on optimal point density & hexagons.



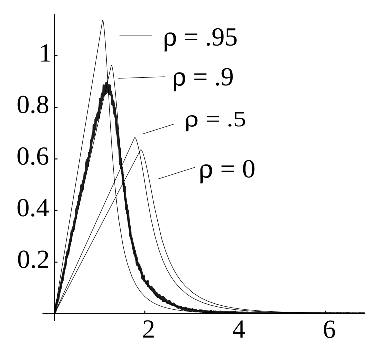
2. Gauss-Markov source with corr. coeff. ρ

   Optimal quantizer designed by LBG algorithm, $\rho = .9$

   k = 2, M = 256, R = 4

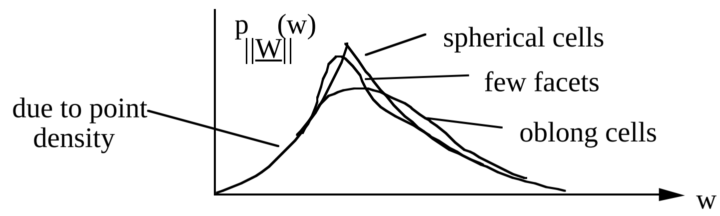
   Predictions based on optimal point density & hexagons.

   Error densities for opt'l quant'rs for Gaussian's differ only by scaling.



PROPERTIES OF ASYMPTOTIC ERROR LENGTH DENSITY

$$p_{\|\underline{W}\|}(w) \cong k V_k w^{k-1} \int p_{\underline{X}}(\underline{x})\, \lambda(\underline{x})\, g\left(\underline{x}, w\lambda(\underline{x})^{1/k}\right) d\underline{x}\,, \quad w \geq 0$$



- Initially rising portion depends on λ, but not g

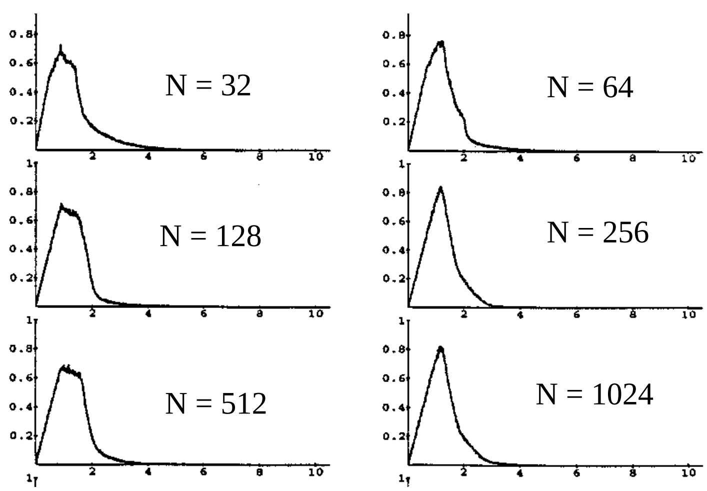
$$p_{\|\underline{W}\|}(w) = \alpha\, w^{k-1} \int \lambda(\underline{x})\, p_{\underline{X}}(\underline{x})\, d\underline{x} \qquad \text{small } w$$

(For small w, the "spheres" are entirely contained in their cells.)

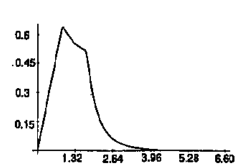
- Location and sharpness of the peak depend primarily on the shapes of the cells.
- An "Experimental" Approach to VQ Analysis
  - + Plot histogram of error length
  - + Use above formula to determine what point densities and cell shapes would cause such.

HISTOGRAMS OF ERROR LENGTH FOR TSVQ

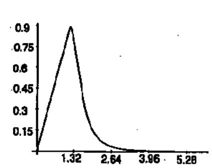
$k = 2$, AR Gaussian source, $\rho = .9$



Measured histograms



1×2 Rectangle

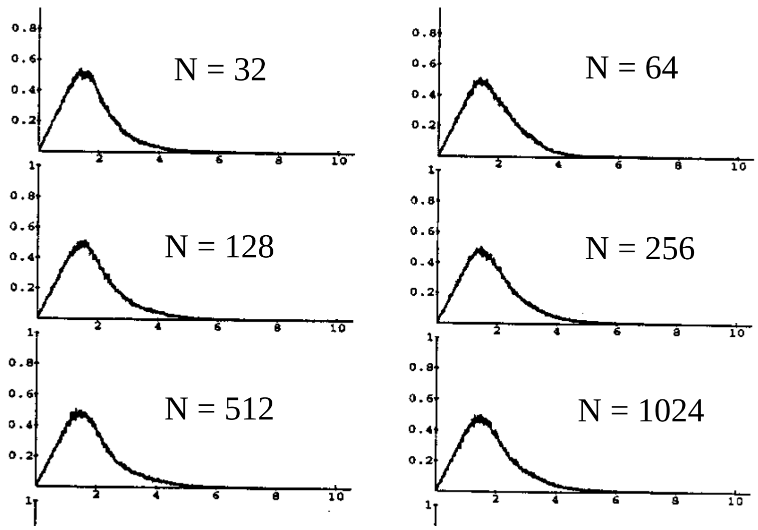


Square

Predictions assuming optimum point density and rectangular or square cells.

HISTOGRAMS OF ERROR LENGTH FOR TSVQ

k = 2,   IID Gaussian Source



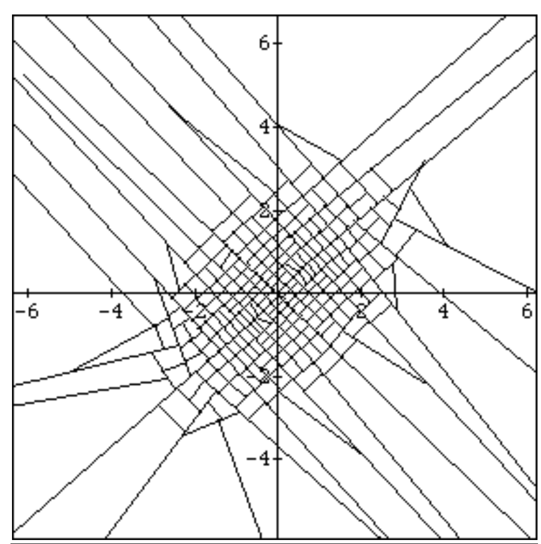
WHAT'S HAPPENING?

For the AR case, at one level, many cells are approximately square, and at the next they are approximately 1×2 rectangles

k = 2

M = 256

R = 4



CONCLUSIONS

- The error histograms:
  - + Support the hypotheses that TSVQ has an optimum point density.
  - + Reflect the distribution of cell shapes; e.g. an equal or unequal mixture of cubes, half-cubes, etc.
- Plotting histograms of the error length gives insight into the geometric structure of a VQ.

ERROR VECTOR DENSITY

(as opposed to Error Length Density)

- Recall $\underline{W} = N^{1/k}\big(\underline{X} - Q(\underline{X})\big)$
- Case 1: Lattice Quantizer

  k-dimensional, small fundamental cell T, sufficiently large support region,

  $$p_{\underline{W}}(\underline{w}) \cong \begin{cases} \lambda , & \underline{v} \in \lambda^{-1/k}\tilde{T} \\ 0 , & \text{otherwise} \end{cases}$$

  where $\lambda \cong 1/N\,\mathrm{vol}(T)$ = point density; $\tilde{T}$ is normalized T

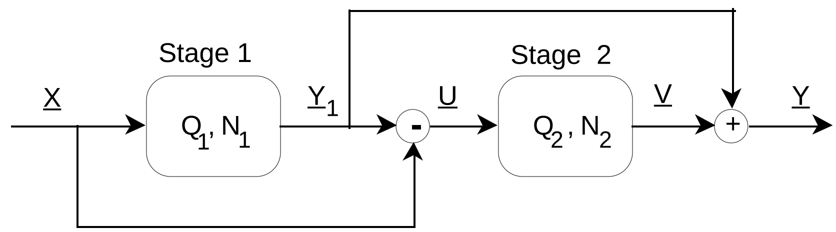
- Case 2: Spherically Symmetric Error Density

  For k-dimensional VQ, with mostly small cells, large N, neighboring cells having similar sizes and shapes, point density $\lambda(\underline{x})$, shape profile $g(\underline{x},r)$, and spherically symmetric error density

  $$p_{\underline{W}}(\underline{w}) = \frac{1}{k V_k ||\underline{w}||^{k-1}} p_{||\underline{W}||}(||\underline{w}||) = \int p_{\underline{X}}(\underline{x})\lambda(\underline{x})\, g(\underline{x}, ||\underline{w}||\lambda(\underline{x})^{1/k})\, d\underline{x}$$

- Conjecture: For large N, opt'l quantizers have, approx'ly, <u>isotropically oriented cells</u> (i.e. among cells of a given size/shape there are roughly equal numbers at all orientations) and, consequently, spherically symmetric error density.
- Alternatively, one may interpret the above as the density of the error assuming a random rotation of the VQ.

BACK TO TWO-STAGE VQ



- Apply Bennett's integral to Stage 2:

$$p_{\underline{U}}(\underline{u}) \;=\; N_1 p_{\underline{W}}\left(\underline{u} N_1^{1/k}\right)$$

$$\cong N_1 \int p_{\underline{X}}(\underline{x})\, \lambda_1(\underline{x})\, g(\underline{x}, \|\underline{u}\| (N_1 \lambda_1(\underline{x}))^{1/k})\, d\underline{x}$$

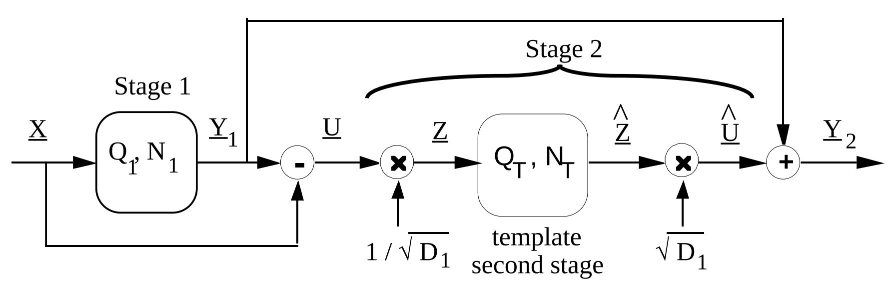
(assuming first-stage has isotropic cells)

$$D = D_2 \cong \frac{1}{N_2^{2/k}} \int \frac{m_2(\underline{u})}{\lambda_2(\underline{u})^{2/k}}\, p_{\underline{U}}(\underline{u})\, d\underline{u}$$

$$= \frac{N_1}{N_2^{2/k}} \int \frac{m_2(\underline{u})}{\lambda_2(\underline{u})^{2/k}} \left(\int p_{\underline{X}}(\underline{x})\, \lambda_1(\underline{x})\, g_1(\underline{x}, \|\underline{u}\| (N_1 \lambda_1(\underline{x}))^{1/k})\, d\underline{x} \right) d\underline{u}$$

- The above formula is not very useful because 2nd-stage point density needs to change with N_1. So, it does not show the real effects of N_1.

NORMALIZE THE SECOND STAGE



- Represent second stage as a scaling of a "template" VQ.

$$Q_T(\underline{z}) = \frac{1}{\sqrt{D_1}} Q_2(\underline{z}\sqrt{D_1}) \qquad \text{or} \qquad Q_2(\underline{z}) = \sqrt{D_1}\, Q_T\!\left(\frac{\underline{z}}{\sqrt{D_1}}\right)$$

$\underline{Z} = \frac{\underline{U}}{\sqrt{D_1}}$ is normalized so that $\frac{1}{k} E\|\underline{Z}\|^2 = 1$

$$\underline{X} - \underline{Y}_2 = \underline{U} - \hat{\underline{U}} = \sqrt{D_1}\,(\underline{Z} - \hat{\underline{Z}})$$

$$D = D_2 = D_1 \times \frac{1}{k} E\,\|\underline{Z} - \hat{\underline{Z}}\|^2 = D_1 D_T$$

- Apply Bennett to D_1 and D_T (assume small cell conditions)

$$D \cong \frac{1}{(N_1N_2)^{2/k}} \int \frac{m_1(\underline{x})}{\lambda_1(\underline{x})^{2/k}} \, p_{\underline{X}}(\underline{x}) \, d\underline{x} \times \int \frac{m_T(\underline{z})}{\lambda_T(\underline{z})^{2/k}} \, p_{\underline{Z}}(\underline{z}) \, d\underline{z}$$

D_S	L
dist'n of N_1N_2 pt. 1-stage with same pt. dens. & inert'l profile as Stage 1	VQ loss factor due to using 2nd stage rather than adding points to Stage 1

where

$$p_{\underline{Z}}(\underline{z}) \cong B_1^{k/2} \int p_{\underline{X}}(\underline{x}) \, \lambda_1(\underline{x}) \, g_1(\underline{x}, ||\underline{z}|| \, B_1^{1/2} \lambda_1(\underline{x})^{1/k}) \, d\underline{x}$$

$$B_1 = \int \frac{m_1(\underline{x})}{\lambda_1(\underline{x})^{2/k}}) \, p(\underline{x}) \, d\underline{x} \quad \text{(assuming isotropic cells)}$$

Conclusions

- $D = D_1 D_T$
- Distortion decreases as 6 dB/bit with R_1 and/or R_2
- All rate allocations give same distortion, if neither R_1 nor R_2 are too small.
- $R_1 = R_2$ for lowest complexity at given distortion and total rate.
- $\lambda_1(x)$, $g_1(x,r)$, $\lambda_T(x)$ and $m_T(x)$ may be optimized independently of N_1 & N_2.
- For multistage VQ: $D = D_S L_2 L_3 \ldots$

GREEDY FIRST STAGE

- Greedy 1st Stage: minimum MSE quantizer with N_1 points (the most common choice). Then loss factor L is ≥ 1
- Greedy 1st stage, optimal opt'l 2nd stage (for 1st-stage errors):

$$L = M_k^* \left(\int p_{\underline{Z}}(\underline{z})^{k/(k+2)} d\underline{z} \right)^{(k+2)/k}$$

$$p_{\underline{Z}}(\underline{z}) \cong (M_k^*)^{k/2} \left(\int p_{\underline{X}}(\underline{x})^{k/(k+2)} d\underline{x} \right)^{(k+1)} \times \int p_{\underline{X}}(\underline{x})^{(k+2)/k} g_1\left(\underline{x}, ||\underline{z}|| \, (M_k^*)^{1/2} A^{-(k+1)/k} p_{\underline{X}}(\underline{x})^{1/(k+2)}\right) d\underline{x}$$

- Gaussian Sources
  - + Greedy First Stage, Optimal Second Stage

    $p_{\underline{Z}}(\underline{z}) = f_k(\underline{z})$ (same for all k-dim'l Gaussian)
  - + Loss factor is the same for all Gaussian sources

k	1	2	4	6
dB loss	6.9	3.0	2.53	1.81

  - + There is a canonical second stage template for Gaussian sources.
- For stationary sources

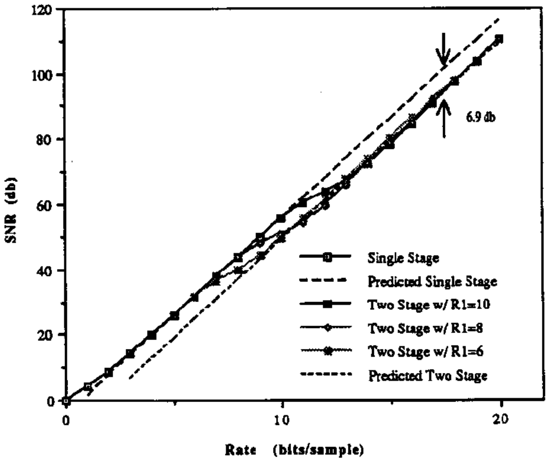
  $L \to 1$ (0 dB) as $k \to \infty$

  (proved by typical sequence arguments).

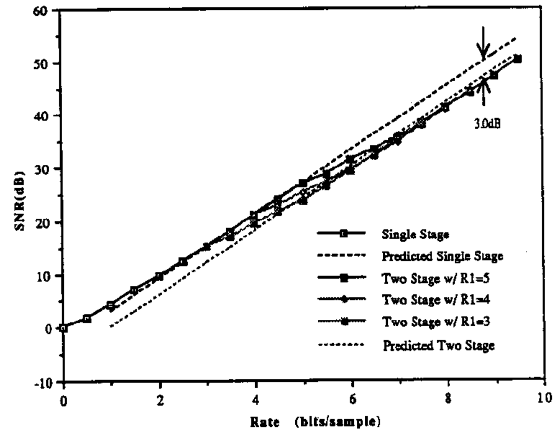
PREDICTED AND ACTUAL SNR

Greedy 2VQ, IID Gaussian Source

k = 1



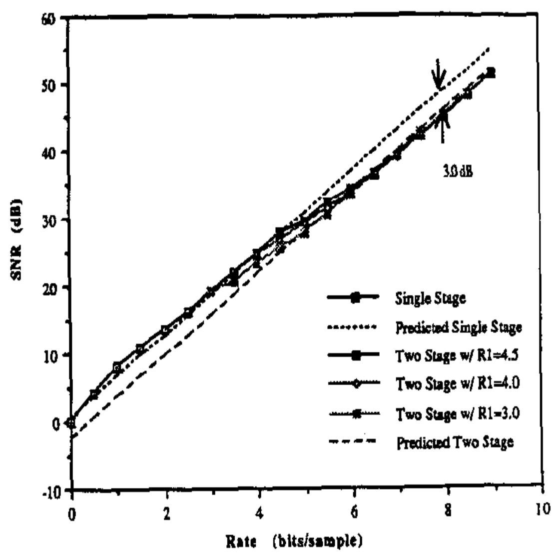
k = 2



PREDICTED AND ACTUAL SNR

Greedy 2VQ
Gaussian AR Source, $\rho = .9$

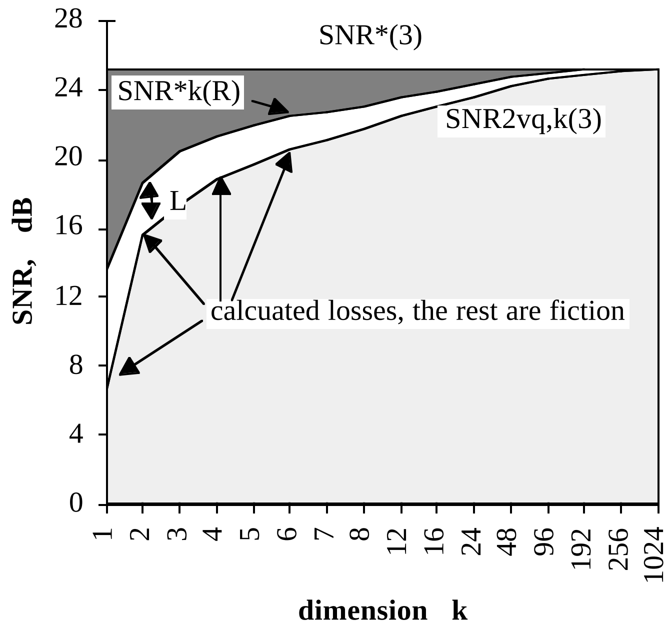
k = 2



EXAMPLE OF PREDICTED SNR FOR 2VQ

- Gaussian AR Source, $\rho = .9$
- $SNR_{tsvq,k}(R) = SNR^*_k(R) - 10 \log_{10} L$

k	1	2	4	6
L dB	6.9	3.0	2.53	1.81



NON-GREEDY FIRST-STAGE

- Non-Greedy First Stage

  + Loss factor L can be less than 1

  Example: k = 2, uniform source, cubic lattice first stage, hexagonal lattice second stage

- Is Greedy Best?

  Conjecture: Optimal first-stage point density should be flatter than the greedy $\lambda(\underline{x}) = c\, p(\underline{x})^{k/(k+2)}$.

  Conjecture: Optimal first-stage inertial profile is conjectured to be C_k, same as greedy.

Summary of Two-Stage Analysis

- Asymptotic analysis of distortion increase of 2VQ over 1VQ.

  $D = D_S L$

- All rate allocations give same distortion.
- SNR increases 6 dB/bit with R_1 and R_2
- For lowest complexity at given rate and distortion, $R_1 = R_2$
- For Gaussian sources, greedy loss factor depends only on dimension.
- For multistage VQ

  $D = D_S L_2 L_3 \ldots$

Variations of Two-Stage VQ

- Can use other types of lossy coders in two stages.
  (e.g. VQ & scalar Q, transform Q and DPCM)
- Multistage quantization: Two or more stages.
- Cell-Conditioned Two-Stage VQ