High Resolution Analysis of Quantizer Distortion

Bennett's Integral

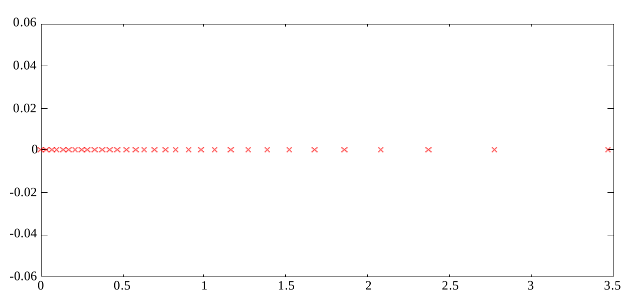
A formula for the mean squared error distortion of a "high-resolution" VQ in terms of its "gross" characteristics.

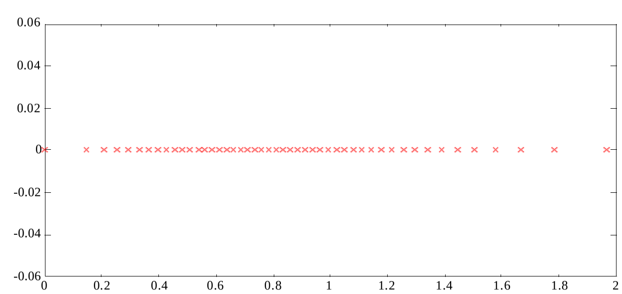
A "high-resolution" VQ is one with "small" cells, so it has "small" distortion and, usually, "many" cells and "large" rate.   Later we'll see how "small", how "many", how "large".

What "gross" characteristics" distinguish different high-resolution quantizers?

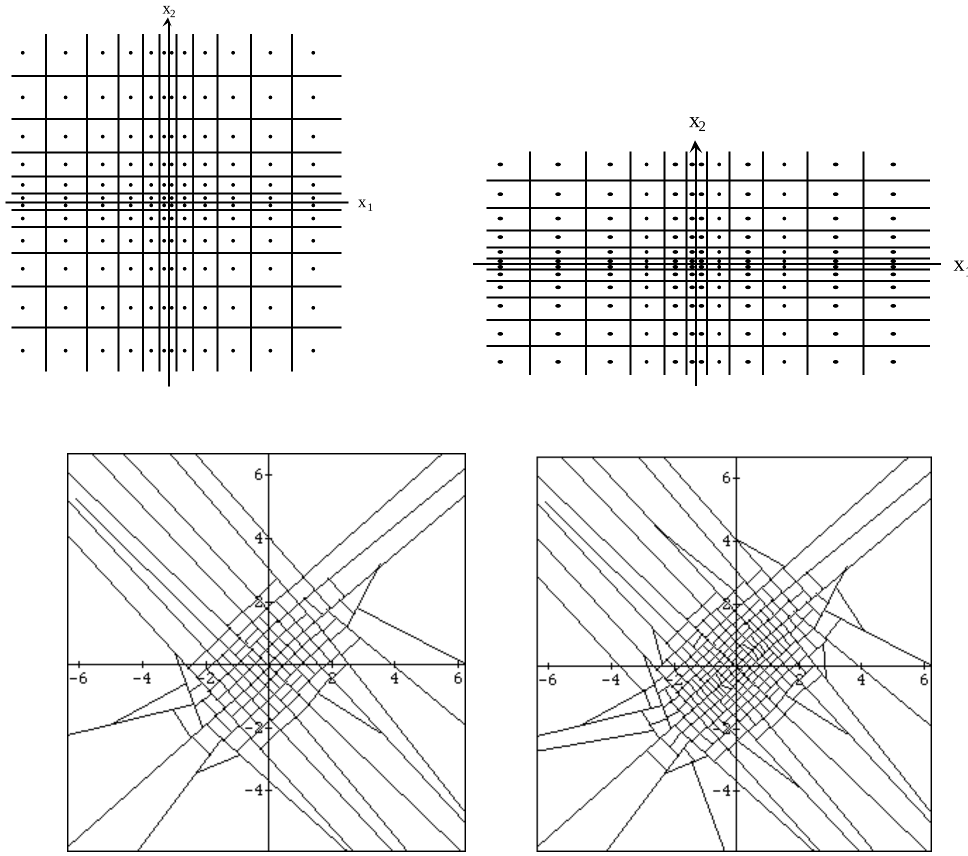
Examples of High-Resolution Quantizers:

k=1 (scalar quantizers)





k=2



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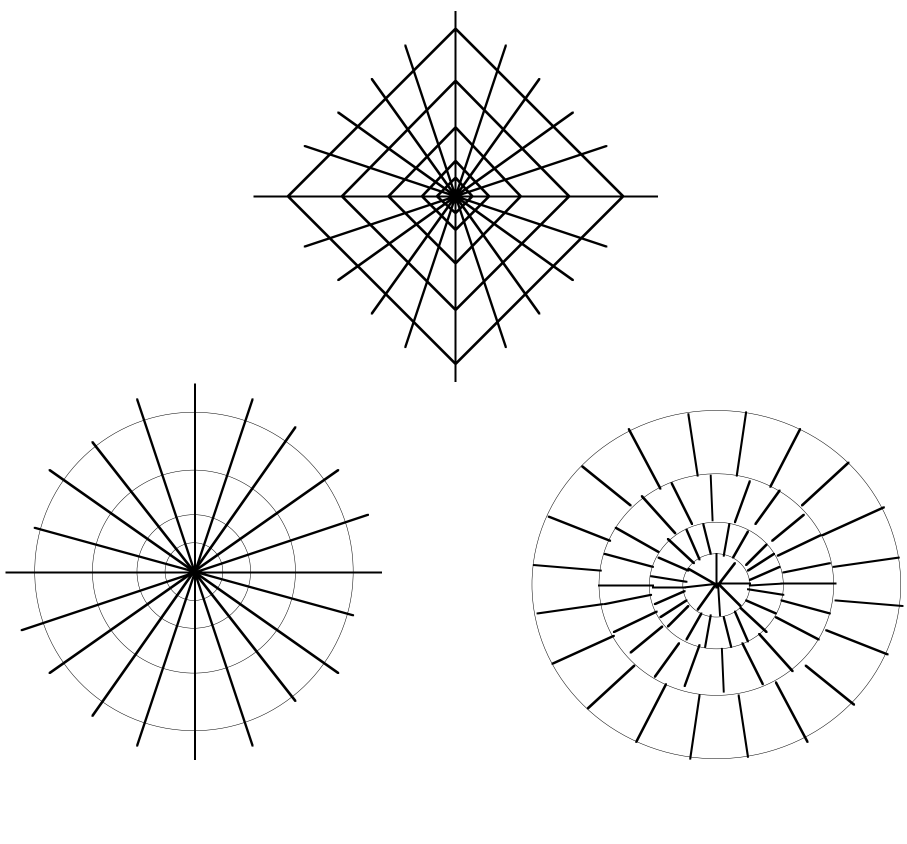

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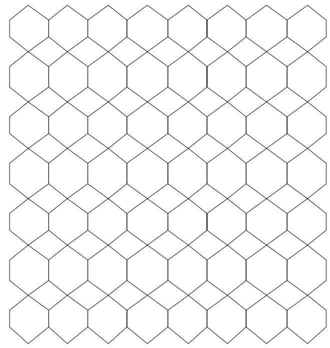
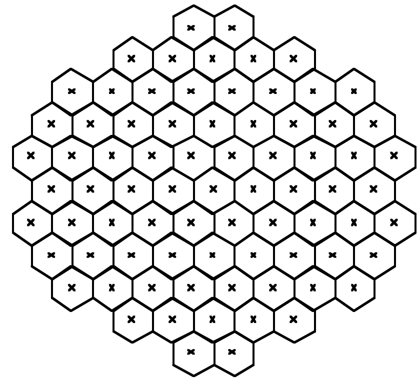
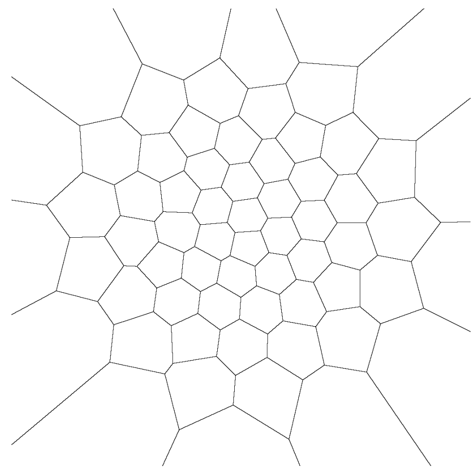






The key gross characteristics are:

- dimension k
- number of points M
- function describing distribution or density of points/cells over R^k
- something to do with the shapes of the cells as a function of position

Case 1: Bennett's Integral for quantizers with congruent cells (Gersho '79)

Under the "high-resolution conditions", the MSE distortion of a k-dimensional VQ with size M can be approximated by

$$D \cong \frac{1}{M^{2/k}} \, m \int \frac{1}{\lambda^{2/k}(\underline{x})} \, f_{\underline{X}}(\underline{x}) \, d\underline{x}$$

where

m = quantity depending only on the shape of the cells

$\lambda(\underline{x})$ = function that characterizes, approximately, the density of points in the vicinity of $\underline{x}$, i.e.

Properties of the Point Density $\lambda(\underline{x})$

1. $\int_A \lambda(\underline{x}) \, d\underline{x} \cong$ fraction of codevectors (or cells) in region A

2. if A is small, then $\lambda(\underline{x}) \, |A| \cong \frac{\#\,\text{points in } A}{M}$

   (assuming A is much larger than cells in the vicinity of $\underline{x}$)

3. $\lambda(\underline{x}) \geq 0, \quad \int \lambda(\underline{x}) \, d\underline{x} = 1$

4. Ordinarily $\lambda(\underline{x})$ is a smooth or piecewise smooth function.

5. $\lambda(\underline{x}) \cong \frac{1}{M|S_i|}$ when $\underline{x} \in S_i$

   Why? In a small region A containing $\underline{x}$, most cells have approximately the same volume. Therefore,

$$\#\,\text{pts in } A \cong \frac{|A|}{\text{cell vol}}$$

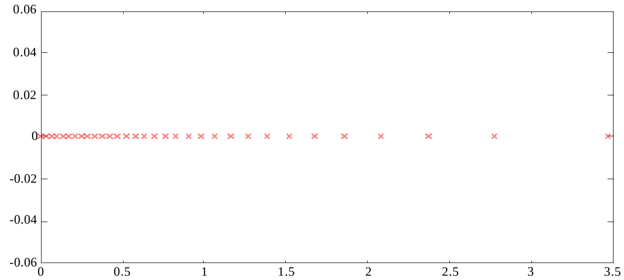
and by 2.

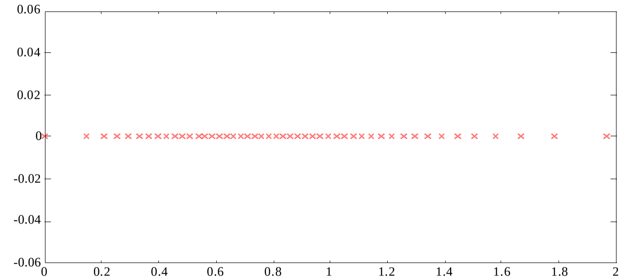
$$\lambda(\underline{x}) \, |A| \cong \frac{\#\,\text{points in } A}{M} = \frac{|A| / (\text{cell vol})}{M}$$

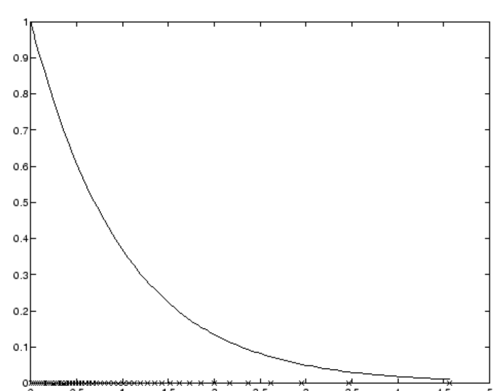
$$\Rightarrow \quad \lambda(\underline{x}) \cong \frac{1}{M|S_i|}$$

Examples of Point Densities

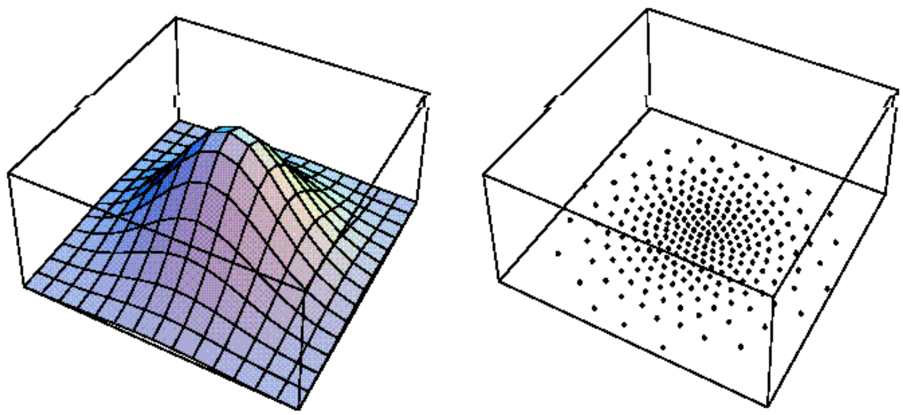
k=1







k=2, optimal quantizer for IID Gaussian vector



Derivation of Bennett's Integral

$$D = \frac{1}{k} E\,||\underline{X}-Q(\underline{X})||^2 = \frac{1}{k}\sum_{i=1}^{M} \int_{S_i} ||\underline{x}-\underline{w}_i||^2 f_{\underline{X}}(\underline{x})\,d\underline{x}$$

$$\cong \frac{1}{k}\sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i) \int_{S_i} ||\underline{x}-\underline{w}_i||^2 d\underline{x} = \sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i)\ \frac{1}{k}\,\mathcal{M}(S_i,\underline{w}_i)$$

where $\mathcal{M}(S,\underline{w}) = \int_S ||\underline{x}-\underline{w}||^2 d\underline{x}$ = "moment of inertia" (mi) of S about $\underline{w}$

Let us separate the effects of shape of S from its size.

$$\frac{1}{k}\,\mathcal{M}(S,\underline{w}) = \frac{\int_S ||\underline{x}-\underline{w}||^2\,d\underline{x}}{k|S|^{1+2/k}} \times |S|^{1+2/k}$$, where $|S|$ = vol of S = $\int_S 1\ d\underline{x}$

$$= m(S,\underline{w}) \times |S|^{1+2/k}$$

where $m(S,\underline{w}) = \frac{\int_S ||\underline{x}-\underline{w}||^2\,d\underline{x}}{k|S|^{1+2/k}}$ = "normalized mom. of inertia" (nmi) of S abt $\underline{w}$.

Fact: $m(S,\underline{w})$ is not affected by a scaling, i.e. it is determined by shape not size.

Consider scaling by a factor a>0: $S \to aS = \{\underline{z} = a\underline{x} : \underline{x} \in S\}$; $\underline{w} \to a\underline{w}$

$$m(aS,a\underline{w}) = \frac{\int_{aS} ||\underline{x}-a\underline{w}||^2 \, d\underline{x}}{k|aS|^{1+2/k}} = \frac{\int_S ||a\underline{z}-a\underline{w}||^2 a^k d\underline{z}}{k|aS|^{1+2/k}}$$ where $a\underline{z} = \underline{x}$, $a^k d\underline{z} = d\underline{x}$

$$= \frac{\int_S a^2 ||\underline{z}-\underline{w}||^2 a^k d\underline{z}}{k|S|^{1+2/k}(a^k)^{1+2/k}}$$ because $|aS| = a^k |S|$

$$= \frac{\int_S ||\underline{z}-\underline{w}||^2 \, d\underline{z}}{k|S|^{1+2/k}} = m(S,\underline{w})$$

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Now back to the formula for distortion

$$D \cong \sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i) \; \frac{1}{k} \; M(S_i,\underline{w}_i) = \sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i) \; m(S_i,\underline{w}_i) \; |S_i|^{1+2/k}$$

$$= m(S_o,\underline{w}_o) \sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i) \; |S_i|^{1+2/k}$$ since all S_i's are congruent to S_o

(need each $\underline{w}_i$ to be in "same position" of S_i as $\underline{w}_o$ is in S_o)

This separately shows the effects of cell size and shape.

Now use the fact that $\lambda(\underline{w}_i) \cong \frac{1}{M|S_i|}$

$$D \cong \frac{1}{M^{2/k}} \; m(S_o,\underline{w}_o) \sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i) \; \frac{1}{\lambda^{2/k}(\underline{w}_i)} \; |S_i|$$

$$\cong \frac{1}{M^{2/k}} \; m(S_o,\underline{w}_o) \; \int f_{\underline{X}}(\underline{x}) \frac{1}{\lambda^{2/k}(\underline{x})} \, d\underline{x}$$

by the definition of an integral and the fact that most $|S_i|$'s are small

which is Bennett's integral.

Later, we'll have more discussion of when it's appropriate to use Bennett's integral, i.e. of the conditions under which it leads to accurate approximations.