Special case: Scalar quantization (k=1)

Cells are intervals.

If codepoints (levels) are in the centers of the cells, then it is easy to show that

$$m(\text{interval}) = \frac{1}{12}$$

Then

$$D \cong \frac{1}{12M^2} \int_{-\infty}^{\infty} \frac{f_X(x)}{\lambda^2(x)} \, dx$$

Special case: Uniform scalar quantizer.

Quantizer is *uniform* over the interval [a,b], in the sense that

(a) the partition divides [a,b] into M cells of width $\Delta = \frac{b-a}{M}$

(c) the codepoints (levels) are in the centers of the cells (i.e. they are uniformly spaced Δ apart)

Suppose that Δ is small, so that

$$\lambda(x) \cong \frac{1}{M\,\Delta} = \frac{1}{a}$$

Suppose also that $Pr(a \le X \le b) \cong 1$. Then

$$D \cong \frac{1}{12M^2} \int_a^b \frac{f_X(x)}{(1/a)^2} \, dx \cong \frac{a^2}{12M^2} \int_a^b f_X(x) \, dx \cong \frac{\Delta^2}{12}$$

This is a formula worth remembering.

Bennett's Integral for Vector Quantizers -- General Case

[Na, Neuhoff, 1995]

Under the *high-resolution conditions* stated below, the MSE distortion of a k-dimensional VQ with size M applied to rand. vector $\underline{X}$ can be approximated by

$$D \cong \frac{1}{M^{2/k}} \int \frac{m(\underline{x})}{\lambda^{2/k}(\underline{x})} f_{\underline{X}}(\underline{x})\, d\underline{x}$$

High-resolution conditions:

+ Most cells are small enough that the prob. density can be approximated as being constant on each.
  (The union of the cells for which the prob. density cannot be so approximated has very small probability. The *overload distortion* is negligible. M is large.)
+ Neighboring cells have similar sizes and shapes.
  (Cell size and shape change slowly, if at all.)
+ The point density is approximately $\lambda(\underline{x})$.
+ The *inertial profile* is approximately $m(\underline{x})$
+ $f_{\underline{X}}(\underline{x})$ = k-dimensional source probability density• For k-dimensional VQ with

Inertial profile: A function $m(\underline{x})$ is a valid inertial profile for a given VQ if

$$m(\underline{x}) \cong \text{NMI of cell containing } \underline{x} = m(S_i,\underline{w}_i) \text{ if } \underline{x} \in S_i$$

We also require $m(\underline{x}) \geq 0$, all $\underline{x}$

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Derivation of Bennett's Integral -- General Case

$$D = \frac{1}{k} E\,||\underline{X}-Q(\underline{X})||^2 = \frac{1}{k}\sum_{i=1}^{M} \int_{S_i} ||\underline{x}-\underline{w}_i||^2 f_{\underline{X}}(\underline{x})\, d\underline{x}$$

$$\cong \frac{1}{k}\sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i) \int_{S_i} ||\underline{x}-\underline{w}_i||^2 d\underline{x} = \sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i)\, \frac{1}{k} M(S_i,\underline{w}_i)$$

MI of S_i about $\underline{w}_i$

$$= \sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i)\, m(S_i,\underline{w}_i)\, |S_i|^{1+2/k}, \qquad m(S_i,\underline{w}_i) = \frac{M(S_i,\underline{w}_i)}{|S_i|^{1+2/k}} = \text{NMI}$$

$$\cong \frac{1}{M^{2/k}} \sum_{i=1}^{M} f_{\underline{X}}(\underline{w}_i) \frac{m(S_i,\underline{w})}{\lambda^{2/k}(\underline{w}_i)} |S_i|, \qquad \lambda(\underline{w}_i) \cong \frac{1}{M|S_i|}$$

$$\cong \frac{1}{M^{2/k}} \int \frac{m(\underline{x})}{\lambda^{2/k}(\underline{x})} f_{\underline{X}}(\underline{x})\, d\underline{x}$$

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Notes on Bennett's Integral

- Bennett's integral identifies point density and inertial profile as key characteristics of VQ's, in addition to k and M. For example, with quantizers with the same k, M and point density, distortion is proportional to the the NMI of the cells.
- When M is large, both left and righthand sides are approximately zero; so what really needs to be shown, and what is really true is

$$\frac{D}{\frac{1}{M^{2/k}} \int \frac{m(\underline{x})}{\lambda^{2/k}(\underline{x})} f_{\underline{X}}(\underline{x})\, d\underline{x}} \cong 1$$

- Bennett's integral shows that distortion decreases as $\frac{1}{M^{2/k}}$,

  assuming point density and inertial profile stay the same.

  To see why, consider what happens when M doubles, while maintaining the same point density function and the same cell shape. Doubling M cuts the volumes of cells in a given region in half. This decreases the linear dimensions of such cells by the factor $1/2^{1/k}$, and causes the average distance squared between points $\underline{x}$ and the codevector of the cell in which $\underline{x}$ lies to decreasesby $1/2^{2/k}$. This indicates distortion decreases as $1/M^{2/k}$.

- Equivalently, SNR increases 6 dB for each one bit increase of rate.

$$\frac{1}{M^{2/k}} = 2^{-2R} \;\Rightarrow\; D \cong 2^{-2R} \int \frac{m(\underline{x})}{\lambda^{2/k}(\underline{x})} f_{\underline{X}}(\underline{x})\, d\underline{x}$$

$$\Rightarrow \text{SNR} = 10 \log_{10} \frac{\sigma^2}{D} = 10 \log_{10} 2^{2R} + 10 \log_{10} \frac{\sigma^2}{\int m(\underline{x})\, \lambda^{-2/k}(\underline{x})\, f_{\underline{X}}(\underline{x})\, d\underline{x}}$$

$$= 6.02\, R + 10 \log_{10} \frac{\sigma^2}{\int m(\underline{x})\, \lambda^{-2/k}(\underline{x})\, f_{\underline{X}}(\underline{x})\, d\underline{x}}$$

- Usually, we don't employ a point density or inertial profile to describe a VQ unless most cells are small, where "small" means that the probability density changes little across the cell and "most" means that the probability of the cells that are small is large.
- Usually, λ and m are fairly smooth functions that does not convey the detailed locations of codevectors and cells. VQ's with the same point density can differ in the number of points, in the exact placement of codepoints and in the shapes of the cells. VQ's with the same inertial profile can differ in the number and placement of codepoints
- Point density and inertial profile are, generally, idealizations or models. Often we pick a target point density or target inertial profile and try to make our quantizer approximate it.

- We don't use the following as definitions because if we did, a quantizer would almost never have a specified point density or inertial profile:

$$\lambda(\underline{x}) = \frac{1}{M|S_i|} \text{ when } \underline{x} \in S_i , \quad m(\underline{x}) = m(S_i,\underline{w}_i) \text{ if } \underline{x} \in S_i$$

- Sketch of why Property 5 on p. 8 implies Property 1

$$\int_A \lambda(\underline{x})\, d\underline{x} = \sum_{i=1}^{M} \int_{S_i \cap A} \lambda(\underline{x})\, d\underline{x} \cong \sum_{i=1}^{M} \int_{S_i \cap A} \frac{1}{M\,\mathrm{vol}(S_i)}\, d\underline{x}$$

$$= \frac{1}{M} \sum_{i=1}^{k} \frac{\mathrm{vol}(S_i \cap A)}{\mathrm{vol}(S_i)} \cong \frac{\#\text{cells in } A}{M}$$

because $\frac{\mathrm{vol}(S_i \cap A)}{\mathrm{vol}(S_i)} = \begin{cases} 1 , \text{ if } S_i \subset A \\ 0 , \text{ if } S_i \cap A = \phi \\ <1 , \text{ otherwise} \end{cases}$, and because most S_i's are small

- When, as usual, $\lambda(x)$ is smooth, Prop. 5, p. 8 implies neighboring cells have similar sizes; e.g. it rules out quantizers with cell sizes with alternating large and small cells.
- When, as usual, m(x) is smooth, the defining property of m implies neighboring cells mostly have similar NMI; e.g. it rules out quantizers whose cell shapes change rapidly.
- A good VQ has larger point density where $f_{\underline{X}}(\underline{x})$ is larger. Small inertial profile is desired, everywhere.

Properties and Examples of Normalized Moment of Inertia (NMI)

- **Definition:** NMI of S about point $\underline{w}$ is

$$m(S) = m(S,\underline{w}) = \frac{1}{k} \frac{\int_S ||\underline{x}-\underline{w}||^2 \, d\underline{x}}{\mathrm{vol}^{1+2/k}(S)}$$

- NMI is the same for cubes of all dimensions.

  This is why the definition of $m(S)$ includes $1/k$.

  k-dimen'l cube $= \{\underline{x} : a \le x_i \le b,\ i = 1,\dots,k\}$ for some $a < b$

  Proof: Let $S = \{\underline{x} : -1/2 \le x_i \le 1/2,\ i = 1,\dots,k\}$ and $\underline{w} = (0,\dots,0)$. Then $\mathrm{vol}(S) = 1$ and

$$m(S) = \frac{1}{k}\frac{1}{\mathrm{vol}(S)^{1+2/k}} \int_{-1/2}^{1/2}\cdots\int_{-1/2}^{1/2} \sum_{i=1}^{k} x_i^2 \, dx_1 \dots dx_k$$

$$= \frac{1}{k} \int_{-1/2}^{1/2}\cdots\int_{-1/2}^{1/2} \left(\sum_{i=2}^{k} x_i^2 + \left.\frac{x_1^3}{3}\right|_{-1/2}^{1/2} \right) dx_2 \dots dx_k$$

$$= \frac{1}{k} \int_{-1/2}^{1/2}\cdots\int_{-1/2}^{1/2} \left(\sum_{i=2}^{k} x_i^2 + \frac{1}{12} \right) dx_2 \dots dx_k$$

$$= \frac{1}{k} \int_{-1/2}^{1/2}\cdots\int_{-1/2}^{1/2} \left(\sum_{i=3}^{k} x_i^2 + \left.\frac{x_2^3}{3}\right|_{-1/2}^{1/2} + \frac{1}{12} \right) dx_3 \dots dx_k$$

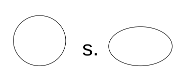
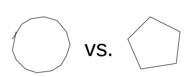
$$= \frac{1}{k} \int_{-1/2}^{1/2}\cdots\int_{-1/2}^{1/2} \left(\sum_{i=4}^{k} x_i^2 + \left.\frac{x_3^3}{3}\right|_{-1/2}^{1/2} + \frac{1}{12} + \frac{1}{12} \right) dx_4 \dots dx_k$$

$$= \frac{1}{k} \sum_{i=1}^{k} \frac{1}{12} = \frac{1}{12}$$

- Examples

cell shape	dimension	NMI
1×2 rectangle	2	.104
cube	any	.0833 = $\frac{1}{12}$
hexagon	2	.0802 = $\frac{5\sqrt{3}}{108}$
circle	2	.0796 = $\frac{1}{4\pi}$
sphere	3	.0770 = $\frac{(4\pi/3)^{-2/3}}{5}$
sphere	k	$\frac{1}{(k+2)(V_k)^{2/k}}$
sphere	∞	.0585 = $\frac{1}{2\pi e} \cong \frac{1}{17}$
$s_1 \times s_2 \times \ldots \times s_k$ rectangle	k	$\frac{1}{12} \frac{\frac{1}{k}\Sigma_{i=1}^{k}(s_i)^2}{\left(\Pi_{i=1}^{k}(s_i)^2\right)^{1/k}} = \frac{1}{12} \frac{\text{arith mean of sides}^2}{\text{geom mean of sides}^2}$

where V_k = volume of k-dimensional sphere with radius 1

- Shapes that tend to make NMI smaller
  + Spheroidal rather than oblong 
  + More finely faceted (many sides rather than few) 
  + Higher rather than lower dimension
- Spheres have the lowest NMI of cells of a given dimension.
- NMI of a sphere decreases with dimension to the limit $1/2\pi e = .0585$

Volume of k-dimensional sphere

(Wozencraft & Jacobs, p. 357)

V_k = vol. of k-dim. sphere with radius 1

$$= \begin{cases} \dfrac{\pi^{k/2}}{(k/2)!}, & k \text{ even} \\[2ex] \dfrac{2^k \pi^{(k-1)/2} \left(\frac{k-1}{2}\right)!}{k!}, & k \text{ odd} \end{cases}$$

From Stirling's approximation,

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\varepsilon_N} \cong \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \text{where } 0 < \varepsilon_n < \frac{1}{12n},$$

(this somewhat underestimates n!),

one can show

$$V_k \cong \frac{1}{\sqrt{\pi k}}\left(\frac{2\pi e}{k}\right)^{k/2}.$$

From the above,

$$V_k \cong \exp\{\frac{k}{2}(\ln(2\pi e) - \ln(k)\}$$

$$\rightarrow 0 \text{ as } k \rightarrow \infty \quad \text{because } -k \ln k \rightarrow -\infty$$

It follows that

$$m(\text{k-dim'l sphere}) = \frac{1}{(k+2)(V_k)^{2/k}} \cong \frac{1}{k+2}\left(\frac{1}{\sqrt{\pi k}}\left(\frac{2\pi e}{k}\right)^{k/2}\right)^{-2/k}$$

$$= (\pi k)^{1/k}\frac{k}{k+2}\frac{1}{2\pi e} = \exp\{\frac{1}{k}\ln(\pi k)\}\frac{k}{k+2}\frac{1}{2\pi e}$$

$$\rightarrow \frac{1}{2\pi e} \text{ as } k \rightarrow \infty.$$

Formal Statement of the Validity of Bennett's Integral

Theorem: $\lim_{M \to \infty} M^{2/k} D(Q_M) = \int \frac{m(\underline{x})}{\lambda^{2/k}(\underline{x})} f_{\underline{X}}(\underline{x})\, d\underline{x}$

Assuming

+ $Q_1, Q_2, \ldots$ is sequence of k-dim'l VQ's, with Q_M having size M, partition S_M
+ $\lambda_M(\underline{x}) \to \lambda(\underline{x})$ in probability as $M \to \infty$

  where $\lambda_M(\underline{x}) \triangleq \frac{1}{M \text{ vol(cell containing } \underline{x})}$ = *specific point denstiy*
+ $m_M(\underline{x}) \to m(\underline{x})$ as $M \to \infty$ in prob.

  where $m_M(\underline{x}) \triangleq m(S_{\underline{x}}, \underline{y}_{\underline{x}})$ = *specific inertial profile*
+ $\{M^{2/k} ||\underline{x} - Q_M(\underline{x})||^2\}$ has uniformlw absolutelw continous integrals
+ diam(cell of S_M containing $\underline{X}$) $\to 0$ in prob.
+ $f_{\underline{X}}(\underline{x})$ is piecewise continuous
+ Bennett's integral is finite

"Sequence approach" first used bw Bucklew & Wise (1982), for scalar quantizers.

Example: Conventional Polar Quantization

Quantize 2-dim'l vector $\underline{X} = (X_1, X_2)$ by independently scalar quantizing its magnitude and phase:

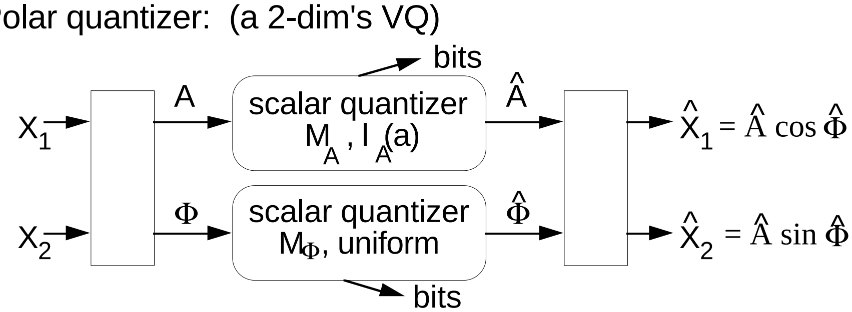
$A = ||\underline{X}|| = \sqrt{X_1^2 + X_2^2}$ = magnitude or amplitude, $A \geq 0$

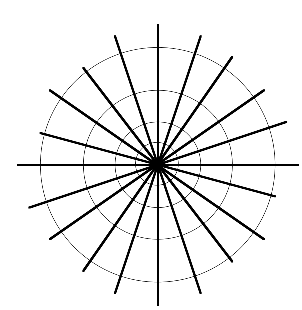
$\Phi = \angle \underline{X} = \tan^{-1}\left(\frac{X_2}{X_1}\right)$ = phase or angle, $0 \leq \Phi \leq 2\pi$

Magnitude quantizer: $\hat{A} = Q_A(A)$, $C_A = \{w_1, \ldots, w_{M_A}\}$, $S_A = \{S_1, \ldots, S_{M_A}\}$

Phase quantizer: $\hat{\Phi} = Q_\Phi(\Phi)$, $C_\Phi = \{v_1, \ldots, v_{M_\Delta}\}$, $T_\Phi = \{T_1, \ldots, T_{M_\Phi}\}$

Polar quantizer: (a 2-dim's VQ)





Size: $M = M_A \times M_\Phi$

Codebook: $C = \{ \underline{z}_{i,j} \}$, where $\underline{z}_{i,j} = (w_i \cos v_j, w_i \sin v_j)$, $i = 1, \ldots, M_A$, $j = 1, \ldots, M_\Phi$

Partition: $S = \{ S_{i,j} \}$, where $S_{i,j} = \{ \underline{x} : ||\underline{x}|| \in S_i, \angle \underline{X} \in T_j \}$, $i = 1, \ldots, M_A$, $j = 1, \ldots, M_\Phi$

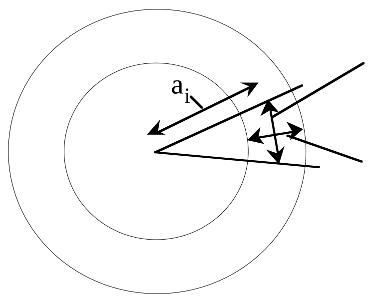
Distortion analysis via Bennett's integral

Assumptions: M_A, M_Φ are large,

magnitude quantizer has point density $\lambda_A(a)$,

phase quantizer is uniform with step size $\Delta = \frac{2\pi}{M_\Phi}$

Then: 2-dim'l polar quant cells are, approximately, small rectangles:



$$W \cong \frac{2\pi a_i}{M_\Phi}$$

$$H \cong \frac{1}{M_A \lambda_A(a_i)}$$

Volume of cell near $\underline{x}$: $HW \cong \frac{2\pi\|\underline{x}\|}{M_A M_\Phi \lambda_A(\|\underline{x}\|)}$

Point density: $\lambda(\underline{x}) = \frac{1}{MHW} = \frac{\lambda_A(\|\underline{x}\|)}{2\pi\|\underline{x}\|}$

Inertial profile:

$$m(\underline{x}) = \frac{1}{12}\frac{\frac{1}{2}(H^2+W^2)}{\sqrt{H^2W^2}} = \frac{1}{24}\left(\frac{M_\Phi}{2\pi\|\underline{x}\| M_A \lambda_A(\|\underline{x}\|)} + \frac{2\pi\|\underline{x}\| M_A \lambda_A(\|\underline{x}\|)}{M_\Phi}\right)$$

Substituting: $\lambda(\underline{x})$ and $m(\underline{x})$ into Bennett's integral and simplifying gives:

$$D \cong \frac{1}{M}\int \frac{m(\underline{x})}{\lambda(\underline{x})} f_{\underline{X}}(\underline{x})\, d\underline{x}$$

$$= \frac{1}{M}\frac{1}{24}\left(\frac{\sqrt{M}}{M_A}\right)^2 \int_0^\infty \frac{1}{\lambda_A(a)^2} p_A(a)\,da + \frac{1}{M}\frac{\pi^2}{6}\left(\frac{M_A}{\sqrt{M}}\right)^2 \int_0^\infty a^2 p_A(a)\,da$$

$$= \frac{1}{M}\frac{1}{L^2}\int_0^\infty \frac{1}{\lambda_A(a)^2} p_A(a)\,da + \frac{1}{M}\frac{\pi^2}{6} L^2 \int_0^\infty a^2 p_A(a)\,da$$

where

$$L \triangleq \frac{M_A}{\sqrt{M}} = \sqrt{\frac{M_A}{M_\Phi}} = \text{ magnitude level allocation}$$

(Recall: $M = M_A \times M_\Phi$)

Optimizing Polar Quantization

For given M (large), choose the key characteristics L and λ_A to minimize distortion

$$D = \frac{1}{M}\frac{1}{L^2}\int_0^\infty \frac{1}{\lambda_A(a)^2}\, p_A(a)\,da + \frac{1}{M}\frac{\pi^2}{6} L^2 \int_0^\infty a^2 p_A(a)\,da$$

Approach 1: For given choice of λ_A, find best L by equating to zero the derivative of D with respect to L. Then find best λ_A by calculus of variations.

Approach 2: For given choice of L, find best λ_A by calculus of variations or Holder's inequality. Then find best L by equating to zero the derivative wrt L of the resulting expression for distortion.

The results

$$\lambda_A(a) = c\, p_A(a)^{1/3} \qquad \text{(c chosen to make } \lambda_A(a) \text{ integrate to one)}$$

$$L^2 = \frac{M_A}{M_\Phi} = \frac{1}{2\pi}\left(EA^2 \int_0^\infty f_A^{1/3}(a)\,da\right)^{-1/2}$$

$$D \cong \frac{1}{12}\, 2\pi \sqrt{E\ A^2}\left(\int_0^\infty f_A^{1/3}(a)\,da\right)^{3/2} \frac{1}{M}$$

Polar Quantization for IID Gaussian

Polar quantization is especially suited to quantizing $\underline{X}$ when $f_{\underline{X}}(\underline{x})$ is circularly symmetric; i.e. when

$f_{\underline{X}}(\underline{x})$ depends only on $||\underline{x}||$.

equivalently,

A and Φ are indep., and Φ is uniformly distributed between 0 and 2π.

equivalently,

$$f_{\underline{X}}(\underline{x}) = \frac{1}{2\pi}\frac{1}{||\underline{x}||} f_A(||\underline{x}||), \qquad \text{where } f_A(a) \text{ is pdf of amplitude } a$$

Why? $f_A(||\underline{x}||)\,\Delta \cong \Pr(||\underline{x}|| \le A \le ||\underline{x}||+\Delta) = \Pr(||\underline{x}|| \le ||\underline{X}|| \le ||\underline{x}||+\Delta) \cong 2\pi\underline{x}\,\Delta\, f_{\underline{X}}(\underline{x})$

Prime Example: X_1, X_2 IID Gaussian

$$f_{\underline{X}}(\underline{x}) = \frac{1}{2\pi} e^{-||\underline{x}||^2/2} = \frac{1}{2\pi ||\underline{x}||}\ ||x||\, e^{-||\underline{x}||^2/2}$$

A has Rayleigh density: $f_A(a) = a\, e^{-a^2/2},\ a \ge 0$

$$EA = \sqrt{\pi/2}, \quad EA^2 = E(X_1^2+X_2^2) = 2\,EX^2 = 2, \quad \sigma_A^2 = 2 - \frac{\pi}{2} = .429$$

For polar quantization optmized for IID Gaussian

$$\lambda_A(a) = c\, p_A(a)^{1/3} = \frac{a}{\sqrt{3}}\, e^{-a^2/6}, \quad a \ge 0$$

$$L^2 = \frac{M_A}{M_\Phi} = \frac{1}{2\pi}\left(EA^2 \int_0^\infty f_A^{1/3}(a)\, da \right)^{-1/2} = .376$$

$$\frac{M_A}{M_\Phi} = .613 \quad \text{(more phase levels than amplitude levels)}$$

$$D \cong \frac{1}{12} \sigma_X^2\, 29.7 \frac{1}{M}$$

Later we'll show that for optimal scalar quantization applies directly to X_1, X_2

$$D \cong \frac{1}{12} \sigma_X^2\, 32.6 \frac{1}{M}$$

Gain of polar quantization over conventional optimal scalar quantization

$$10 \log_{10} \frac{32.6}{29.7} = .41 \text{ dB}$$

Calculus of Variations

Fix M and L. Let $J(\lambda_A)$ be the functional defined by the formula for the distortion of a polar quantizer.

If λ_A is the optimal point density, i.e the one that makes $J(\lambda_A)$ smallest among all nonnegative functions λ that integrate to one, then for any function g such that $\int_0^\infty g(a)\, da = 0$ and any $\varepsilon > 0$, $\lambda_A(a)+\varepsilon g(a)$ cannot be better than $J(\lambda_A)$, i.e.

$$J(\lambda_A+\varepsilon g) \text{ is minimized when } \varepsilon = 0,$$

Therefore,

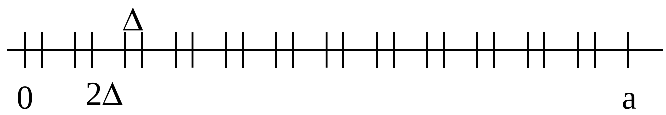
$$\frac{d}{d\varepsilon} J(\bar{\lambda}_A+\varepsilon g)\Big|_{\varepsilon=0} = 0$$

and this must be true for every function g such that $\int_0^\infty g(a)\, da = 0$. Therefore, we find the derivative of $J(\lambda_A+\varepsilon g)$ with respect to ε. We set $\varepsilon = 0$ and we equate the derivative at $\varepsilon=0$ to zero. This gives an equation that λ_A must satisfy for every function g such that $\int_0^\infty g(a)\, da = 0$. Solving this equation gives the answer.

Where Bennett's Integral Does Not Apply

Suppose neighboring cells do not have similar sizes

Example:



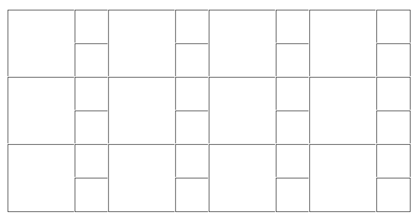
$k = 1, \quad \lambda(x) = \frac{1}{a}, \quad m(x) = \frac{1}{12}, \quad \frac{M}{2}\Delta + \frac{M}{2}2\Delta = a \;\Rightarrow\; \frac{a}{M} = \frac{3}{2}\Delta$

$$\text{Bennett's integral} = \frac{1}{12}\frac{1}{M^2}\int_0^a \frac{1}{1/a^2} f(x)\,dx = \frac{1}{12}\frac{a^2}{M^2} = \frac{1}{12}\frac{9}{4}\Delta^2$$

$$\text{Distortion} \cong \frac{1}{3} D(\Delta) + \frac{2}{3} D(2\Delta) = \frac{1}{3}\frac{\Delta^2}{12} + \frac{2}{3}\frac{(2\Delta)^2}{12} = \frac{1}{12} 3\,\Delta^2$$

Another problem example:

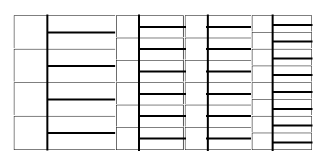
$k = 2$



- Bennett's integral can be applied apply to quantizers where the cell shapes change rapidly (e.g. a periodic tesselation) provided neighboring cells have similar volumes. In this case, we take $m(\underline{x})$ to be the average of the NMI's of the cells in the vicinity of $\underline{x}$.

Example 1:

Bennett's integral can be applied with



$$m(\underline{x}) = \frac{1}{3}\left(\frac{1}{12} + 2\,\frac{1}{12}\,1.25\right)$$

$$\left(\text{NMI}(a \times b \text{ rect'l}) = \frac{1}{12}\frac{(a^2+b^2)/2}{\sqrt{a^2b^2}}\right)$$

Example 2:

Bennett's integral will not give the correct distortion because neighboring cells do not have the identical or similar volumes.