High-Resolution Analysis of Least Distortion of Fixed-Rate Vector Quantizers

Zador's Formula

- Begin with Bennett's Integral

$$D \cong \frac{1}{M^{2/k}} \int \frac{m(\underline{x})}{\lambda^{2/k}(\underline{x})} f_{\underline{X}}(\underline{x})\, d\underline{x}$$

- Find best inertial profile
- Find best point density
- Substitute into Bennett's integral to obtain high-resolution approximation to the OPTA function of k-dimensional VQ.

Best Inertial Profile

- For any dimension k and probability distribution, the best inertial profile is a constant. The constant is independent of the source probability distribution.
- Sketch of the proof: (no rigorous proof exists)
  - Claim 1: When the source random vector is uniformly distributed on a k-dimensional cube, the best inertial profile is a constant. That is the inertial profile of an optimal quantizer is, approximately, a constant.

    Let m_k^* denote the constant.

    (Actually, one can replace "cube" with "sphere" or any other sufficiently nice convex region.)

- Claim 2: For any k-dimensional probability density function, the best inertial profile is the constant m_k^*. That is, an optimal quantizer with many points has $m(\underline{x}) = m_k^*$.

Gersho's Conjecture:

- When M is large, most cells of an optimal, k-dimensional, M-point, fixed-rate quantizer are approximately congruent to some basic cell shape.
- The basic cell shape is the k-dimensional tesselating polyhedron with least NMI.
- In most small regions, the quantizer partition is a "tesselation" based on this basic cell shape.

  A cell "tesselates" or "tiles" if there exists a partition of R^k with each cell being a translation or rotation (but not a scaling) of the basic cell shape.
- m_k^* = least nmi of any tesselating k-dimensional polyhedron

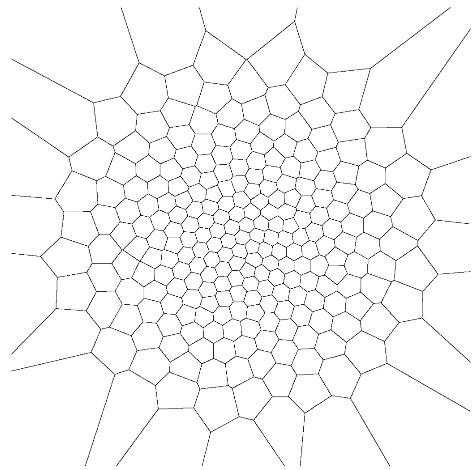
  = Gersho's constant

Notes:

- Since the optimal quantizer will ordinarily have cells of different sizes, its partition will not be a tesselation. However, in small regions the tesselation will be apparent, i.e. locally it is approximately a tesselation.
- The tesselation generated by the basic cell shape might even be a "lattice", meaning that all cells of the tesselation are translations (rather than rotations) of one another.
- We will assume Gersho's conjecture is correct when we need the value of m_k^*. If this is not exactly correct, it is not off by much.

Example:

A two-dimensional VQ with $M = 256$ designed by the LBG algorithm for an IID Gaussian source.



Best Point Density

Assuming the best inertial profile is a constant, we need to find the point density minimizing the remaining part of Bennett's integral.

$$\int \frac{f_{\underline{X}}(\underline{x})}{\lambda^{2/k}(\underline{x})}\, d\underline{x} \qquad (*)$$

It could be found with calculus of variations, but we will use Holder's inequality.

Holder's inequality:

Given function's f and g, then for any $q,r > 0$ such that $\frac{1}{q} + \frac{1}{r} = 1$,

$$\int |f(\underline{x})\, g(\underline{x})|\, d\underline{x} \le \left(\int |f(\underline{x})|^q d\underline{x} \right)^{1/q} \left(\int |g(\underline{x})|^r d\underline{x} \right)^{1/r}$$

with equality iff for some c, $|f(\underline{x})|^q = c\, |g(\underline{x})|^r$, all $\underline{x}$

Strategy: Choose f, g, q and r so that

$$|f(x)|^q = \frac{f_X(\underline{x})}{\lambda(\underline{x})^{2/k}} \quad \text{and} \quad \int |g(\underline{x})|^r dx = 1.$$

Then

$$\int \frac{f_X(\underline{x})}{\lambda(\underline{x})^{2/k}} d\underline{x} = \int |f(\underline{x})|^q d\underline{x} \ge \left(\int |f(\underline{x})\, g(\underline{x})|\, d\underline{x} \right)^q \left(\int |g(\underline{x})|^r d\underline{x} \right)^{-q/r} = \left(\int |f(\underline{x})\, g(\underline{x})|\, d\underline{x} \right)^q$$

with equality iff there is constant c such that $|f(\underline{x})|^q = c\, |g(\underline{x})|^r$.

If it turns out that the last integral does not depend on λ, then we have a lower bound to the integral we are minimizing. And we can minimize the integral by choosing λ to satisfy the condition that gives equality in the lower bound.

Our choices:

$$q = \frac{k+2}{k}, \quad r = \frac{k+2}{2} \qquad \Rightarrow \frac{1}{q} + \frac{1}{r} = \frac{k}{k+2} + \frac{2}{k+2} = 1$$

$$f(\underline{x}) = \left(\frac{f_X(\underline{x})}{\lambda^{2/k}(\underline{x})} \right)^{k/(k+2)} \quad \text{and} \quad g(\underline{x}) = \lambda^{2/(k+2)}(\underline{x})$$

Then as desired, $|f(\underline{x})|^q = \frac{f_X(\underline{x})}{\lambda^{2/k}(\underline{x})}$ and $\left(\int |g(\underline{x})|^r d\underline{x} \right)^{q/r} = \left(\int \lambda(\underline{x})\, d\underline{x} \right)^{q/r} = 1$

Therefore, $\int \frac{f_X(\underline{x})}{\lambda^{2/k}(\underline{x})} d\underline{x} \ge \left(\int |f(\underline{x})\, g(\underline{x})|\, d\underline{x} \right)^q = \left(\int f_{\underline{X}}^{k/(k+2)}(\underline{x})\, d\underline{x} \right)^{(k+2)/k}$

where, fortunately, the right-hand side does not depend on λ, and where equality holds iff there is a constant c such that

$$\frac{f_X(\underline{x})}{\lambda^{2/k}(\underline{x})} = c\, \lambda(\underline{x}), \text{ i.e. } \lambda(\underline{x}) = c'\, f_{\underline{X}}^{k/(k+2)}(\underline{x})$$

where c' is chosen to make $\lambda(\underline{x})$ integrate to one. We conclude that the integral (*) is minimized by the point density

$$\lambda_k^*(\underline{x}) = \frac{f_{\underline{X}}^{k/(k+2)}(\underline{x})}{\int f_{\underline{X}}^{k/(k+2)}(\underline{x}')\, d\underline{x}'}$$

and the resulting minimum value is $\left(\int f_{\underline{X}}^{k/(k+2)}(\underline{x})\, d\underline{x} \right)^{(k+2)/k}$

Substituting λ_k^* and m_k^* into Bennett's integral yields:

Zador's Theorem (Zador, 1963)

When M is large, the least distortion of fixed-rate, k-dim'l VQ with M points is

$$\delta(k,M) \cong \sigma^2 \beta_k\, m_k^* \frac{1}{M^{2/k}} \triangleq Z(k,M)$$

where $Z(k,M)$ is called Zador's function

$$\sigma^2 = \text{source variance} = \frac{1}{k}\sum_{i=1}^{k} \text{variance}(X_i)$$

$$\beta_k = \frac{1}{\sigma^2}\left(\int f_{\underline{X}}(\underline{x})^{k/(k+2)}\, d\underline{x}\right)^{(k+2)/k} = \text{"Zador's factor"}$$

depends on "shape" of $f_{\underline{X}}(\underline{x})$; invariant to a scaling)

m_k^* = least NMI of k-dim'l tesselating polytopes (we assume)

Equivalent Statements

When R is large, the best fixed-rate, k-dimensional VQ's with rate R have MSE

$$\delta(k,R) \cong \sigma^2 \beta_k\, m_k^*\, 2^{-2R} \triangleq Z(k,R)$$

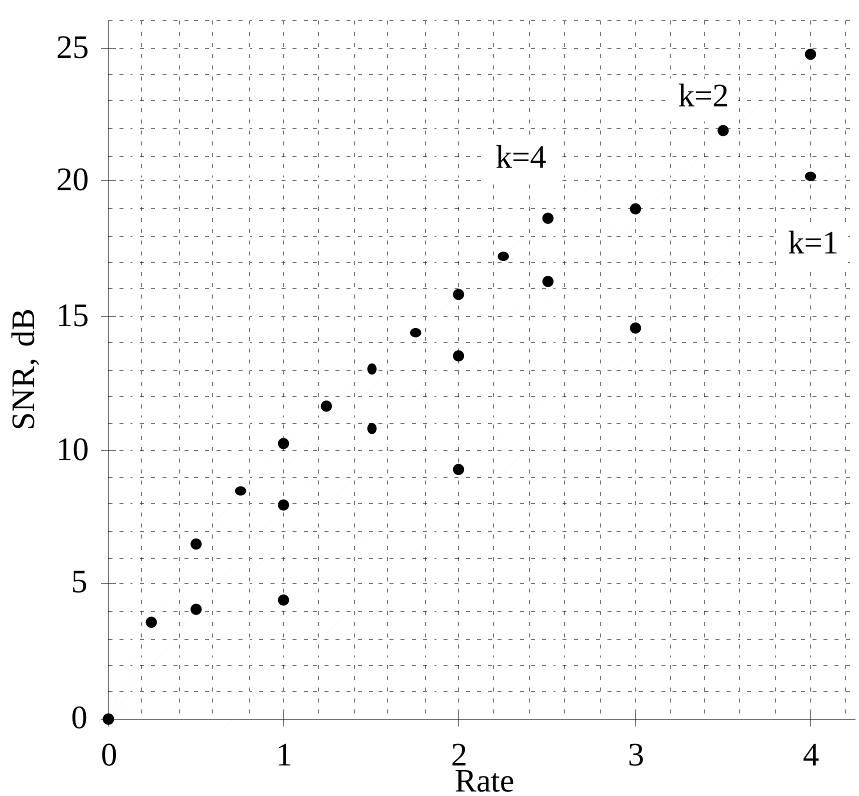
When R is large, the best fixed-rate, k-dimensional VQ's with rate R have SNR

$$S(k,R) \cong 10 \log_{10} \frac{\sigma^2}{Z_k(R)} = 6.02\, R - 10 \log_{10} m_k^* \beta_k$$

Note: SNR increases at 6 dB per bit for optimal quantizers.

How large must M or R be in order for the formulas to be accurate**?**

Example: Gauss-Markov Source, corr. coeff. $\rho = .9$



VQ's designed by LBG algorithm.

Straight lines are Zador's funtion Z(k,R).

m_4^* is estimated.

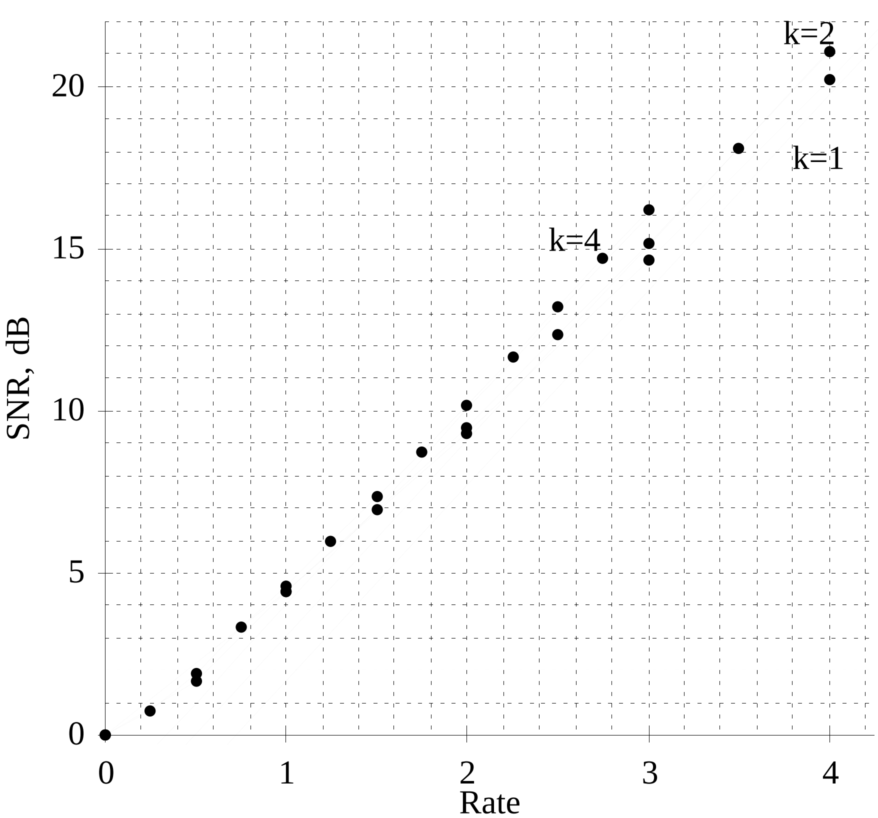
Data for the previous plot

Gauss-Markov Source, $\rho = .9$, SNR's in dB

Rate	k = 1 Act'l	k = 1 Pred'd	k = 1 Diff.	k = 2 Act'l	k = 2 Pred'd	k = 2 Diff.	k = 4 Act'l	k = 4 Pred'd	k = 4 Diff.
0.0	0.0	-4.35	4.35	0.0	0.6	-0.56	0.00	3.33	-3.33
0.5				4.0	3.6	0.47	6.55	6.34	0.21
1.0	1.68	4.4	2.72	7.9	6.6	1.35	10.22	9.35	0.87
1.5				10.8	9.6	1.21	13.04	12.36	0.68
2.0	7.70	9.3	1.60	13.5	12.6	0.92	15.81	15.37	0.44
2.5				16.3	15.6	0.65	18.66	18.38	0.27
3.0	13.72	14.62	0.90	19.0	18.6	0.41			
3.5				21.9	21.6	0.26			
4.0	19.74	20.22	0.48	24.8	24.6	0.16			

The predicted value at $R = 0$ is $-10 \log_{10} m_k^* \beta_k$

Example 2: IID Gaussian Source



VQ's designed by LBG algorithm.

Straight lines are from Zador's function Z(k,R).

m_4^* is estimated.

Data for the previous plot
IID Gaussian Source, SNR's in dB

	k = 1			k = 2			k = 4		
Rate	Act'l	Pred'd	Diff.	Act'l	Pred'd	Diff.	Act'l	Pred'd	Diff.
0.0	0.0	-4.35	4.35	0.00	-2.95	2.95	0.00	-2.01	2.01
0.5				1.66	0.06	1.60	1.89	1.00	0.89
1.0	1.68	4.4	2.72	4.39	3.07	1.32	4.60	4.01	0.60
1.5				6.96	6.08	0.88	7.34	7.02	0.32
2.0	7.70	9.3	1.60	9.64	9.09	0.55	10.18	10.03	0.15
2.5				12.42	12.10	0.32	13.21	13.04	0.17
3.0	13.72	14.62	0.90	15.27	15.11	0.16	16.18	16.05	0.14
3.5				18.17	18.12	0.05			
4.0	19.74	20.22	0.48	21.12	21.13	-0.01			

The predicted value at $R = 0$ is $-10 \log_{10} m_k^* \beta_k$