Rules of Thumb:

- Z-G is accurate for $R \gtrsim 3$ $(M \gtrsim 2^{3k})$.
- For a given R, accuracy increases with dimension k.

Properties of m_k^*

(assuming Gersho's conjecture)

- m_k^* is only known for $k = 1, 2$

  $m_1^* = m(\text{interval}) = \frac{1}{12} = .0833$

  $m_2^* = m(\text{hexagon}) = 5\sqrt{3}/108 = .0802$

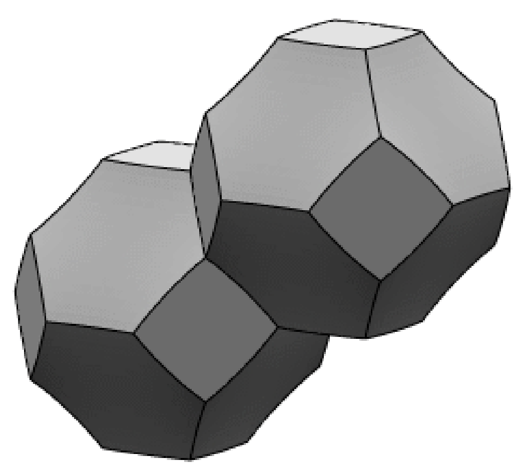
- For k = 3, the best lattice tesselation is known to be that generated by the truncated octahedron, illustrated to the righ for which

  $m = .0785.$

  But it is not known that this is the best tesselation. Hence,

  $m_3^* \leq .0785$

- It is not known if $m_{k+1}^* \leq m_k^*$ for all k.



- Though m_k^* might not be decreasing with k, it can be shown to be "subadditive", meaning that for any k,l

$$m_{k+l}^* \le \frac{k}{k+l} m_k^* + \frac{l}{k+l} m_l^*$$

which implies (with appropriate use of the above)

$$m_1^* \ge m_k^* \ge m_{2k}^*$$

It also can be shown that subadditivity implies

$$m_\infty^* \triangleq \lim_{k\to\infty} m_k^* = \inf_k m_k^*$$

A proof that subadditivity implies lim = inf can be found in Gallager's information theory book, Lemma 2, pp. 112,113.

The proof of subadditivity depends on the following fact:

Fact: If $S \subset R^k$ and $T \subset R^l$, then

$$M(S\times T) = \text{vol}(S)\, M(T) + \text{vol}(T)\, M(S)$$

where $M(S) = \int_S ||\underline{x}||^2 d\underline{x} = MI$

Proof of Fact:

$$M(S\times T) = \int_{S\times T} ||\underline{x}||^2 d\underline{x} = \int_S \int_T (||\underline{x}||^2 + ||\underline{y}||^2\, d\underline{x}\, d\underline{y}$$

$$= \int_S (I(T) + ||\underline{y}||^2 \text{vol}(T))\, d\underline{y} = M(T)\,\text{vol}(S) + M(S)\,\text{vol}(T)$$

Proof of Subadditivity:

Assuming Gersho's conjecture, let S and T be tesselating polyhedra with unit volumes that achieve m_k^* and m_l^*, respectively. Then $S\times T$ is also a tesselating polyhedron. And vol($S\times T$) = 1. Therefore, applying the Fact,

$$m_k^* \le m(S\times T) = \frac{M(S\times T)}{(k+l)\text{vol}(S\times T)^{(k+l+2)/(k+l)}}$$

$$= \frac{M(T)\,\text{vol}(S) + M(S)\,\text{vol}(T)}{k+l} = \frac{1}{k+l}(M(T) + M(S))$$

$$= \frac{1}{k+l}(l\, m(T) + k\, m(S) +)) = \frac{k}{k+l} m_k^* + \frac{l}{k+l} m_l^*$$

- Sphere Lower bound:

$$m_k^* \geq m(\text{k-dim sphere}) = \frac{(V_k)^{-2/k}}{k+2} \rightarrow \frac{1}{2\pi e} = .0585 \cong \frac{1}{17}$$

  where V_k = vol. of k-dim. sphere w radius 1 $\rightarrow \frac{1}{2\pi e}$

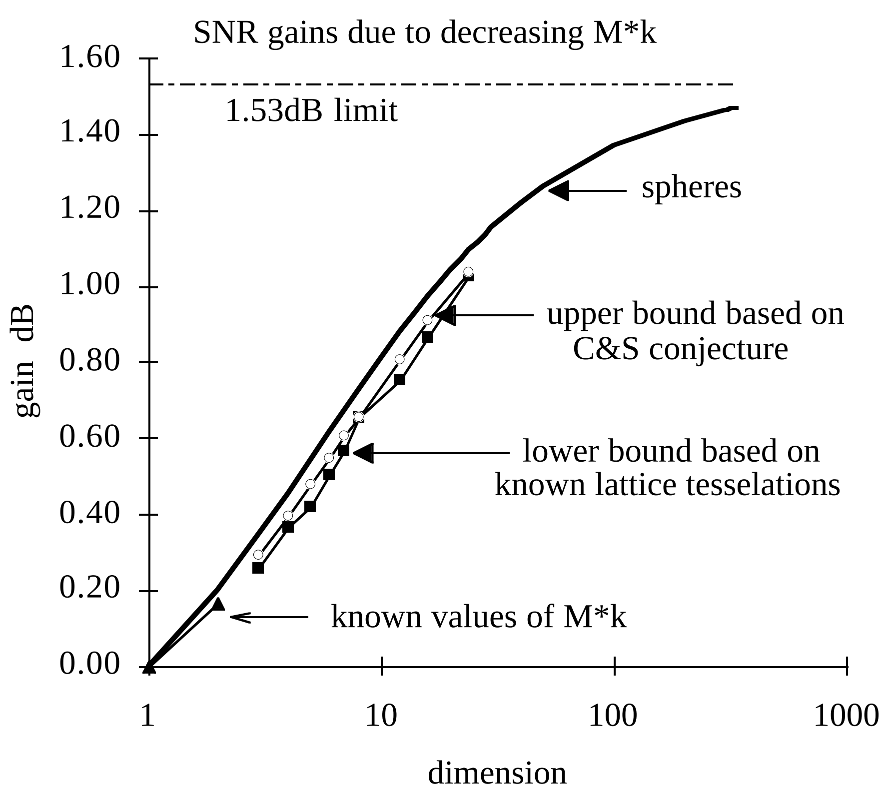
- $m_k^* \rightarrow \frac{1}{2\pi e} = .0585 \cong \frac{1}{17}$ as $k \rightarrow \infty$.
- Upper bounds: $m_k^* \leq m(S)$ for best known tesselating polyhedron. Such bounds may continue to improve as people learn of better tesselating polyhedra.
- There is a conjectured lower bound in Conway and Sloane's book, p. 59-62. It is tighter than the sphere lower bound
- Summary: m_k^* decreases with k (though not neccessarily monotonically) from $1/12 = .0833$ at $k = 1$ to $m_\infty^* = 1/2\pi e = .0585 \cong 1/17$, which represents a gain of 1.53 dB.
- The book by Conway and Sloane has a summary of what is known about the best tesselating polytopes.
- We need a good name for m_k^*. Any suggestions?

The Best Known Tesselating Polytopes

dimension	polytope	m_k^*	best known, (upper bound)	conj'd lower bound	sphere lower bound	gain (dB) $10 \log m_1^*/m_k^*$
1	interval	.0833'			.0833	0
2	hexagon	.0802'			.0796	.16
3	unknown		.0785' truncated octahedron	.077875	.0770	.26
4	"		.0766	0.0761'	.0750	.39
5	"		.0756	0.0747'	.0735	.47
6	"		.0742	0.0735'	.0723	.55
7	"		.0731	0.0725'	.0713	.60
8	"		.0717	0.0716'	.0705	.66
12	"		.0701	0.0692'	.0691	.81
16	"		.0683	0.0676'	.0666	.91
24	"		.0658	0.0656'	.0647	1.10

50	"				.0623'	1.26
100	"				.0608'	1.37
200	"				.0599'	1.43
300	"				.0595'	1.46
very large	sphere	.0585'			.0585	1.53

The gains shown are computed from the primed values.



Properties of Zador's Factor, β_k

(1) If $\underline{Y} = A\underline{X} + \underline{b}$, where A is a nonsingular square matrix, then

$$\beta_{Y,k} = \frac{\sigma_X^2}{\sigma_Y^2} |A|^{2/k} \beta_{X,k}$$

Note that β is not affected by $\underline{b}$.

Derivation: We have

$$f_{\underline{Y}}(\underline{y}) = |A|^{-1} f_{\underline{X}}(A^{-1}(\underline{y}-\underline{b}))$$

Therefore,

$$\beta_{Y,k} = \frac{1}{\sigma_Y^2}\left(\int f_{\underline{Y}}(\underline{y})^{k/k+2}\, d\underline{y}\right)^{(k+2)/k}$$

$$= \frac{1}{\sigma_Y^2}\left(\int (|A|^{-1} f_{\underline{X}}(A^{-1}(\underline{y}-\underline{b})))^{k/k+2}\, d\underline{y}\right)^{(k+2)/k}$$

$$= \frac{1}{\sigma_Y^2}|A|^{-1}\left(\int f_{\underline{X}}(\underline{x})^{k/k+2}\, |A|\, d\underline{x}\right)^{(k+2)/k}, \text{ with } \underline{x} = A^{-1}(\underline{y}-\underline{b}),\ \underline{y}=A\underline{x}+\underline{b},\ d\underline{y} = |A|\, d\underline{x}$$

$$= \frac{1}{\sigma_Y^2}|A|^{2/k}\left(\int f_{\underline{X}}(\underline{x})^{k/k+2}\, d\underline{x}\right)^{(k+2)/k}$$

$$= \frac{1}{\sigma_Y^2}|A|^{2/k}\sigma_X^2\,\beta_{X,k}$$

(2) If $\underline{Y} = a\underline{X} + \underline{b}$, where $a \neq 0$, then $\beta_{Y,k} = \beta_{X,k}$. This shows β_k depends on the shape of the density, and is invariant to scaling or shifting.

Derivation: This can be derived directly, or from Property (1) with A being a diagonal matrix with a's on the diagonal and $|A| = a^k$, and with $\sigma_Y^2 = a^2\sigma_X^2$.

(3) If $\underline{Y} = A\underline{X} + \underline{b}$ and A is a $k \times k$ orthogonal matrix (i.e. $A^{-1} = A^t$), then $\beta_{Y,k} = \beta_{X,k}$.

It should be intuitive that the opta for $\underline{Y}$ is the same as for $\underline{X}$, because one can rotate and translate an optimal VQ for $\underline{X}$ to get a VQ with the same performance for $\underline{Y}$, and vice versa.

Derivation: For an orthogonal matrix $||A\underline{x}|| = ||\underline{x}||$ for all $\underline{x}$. Therefore,

$$\sigma_Y^2 = \frac{1}{k} E||\underline{Y}-E\underline{Y}||^2 = \frac{1}{k} E||A(\underline{X}-E\underline{X})||^2 = \frac{1}{k} E||\underline{X}-E\underline{X}||^2 = \sigma_X^2$$

Next $|A| = \prod_{i=1}^{k} \lambda_i = 1$, where the λ_i's are the eigenvalues, which all have magnitude one, because $A\underline{x} = \lambda\underline{x}$ implies $||\underline{x}|| = ||A\underline{x}|| = ||\lambda\underline{x}|| \Rightarrow |\lambda| = 1$.

(4) If $\underline{Y} = A\underline{X} + \underline{b}$, where A is a $k \times k$ diagonal matrix with diagonal elements $a_1,\ldots,a_k$, then

$$\beta_{Y,k} = \left(\prod_{i=1}^{k} a_i^2\right)^{1/k} \frac{\sum_{i=1}^{k} \sigma_{X,i}^2}{\sum_{i=1}^{k} a_i^2 \sigma_{X,i}^2} \beta_{X,k}$$

$$= \frac{\left(\prod_{i=1}^{k} a_i^2\right)^{1/k}}{\frac{1}{k}\sum_{i=1}^{k} a_i^2} \beta_{X,k} \quad \text{if } \sigma_{X,i}^2\text{'s are all the smae}$$

(5) If $X_1,\ldots,X_k$ are independent, then

$$\beta_k = \frac{1}{\sigma_X^2} \prod_{i=1}^{k} \left(\int f_i(x)^{k/(k+2)} dx\right)^{(k+2)/k}$$

Derivation:

$$\beta_{X,k} = \frac{1}{\sigma_X^2} \left(\int f_1(x_1)^{k/k+2} \ldots f_k(x_k)^{k/k+2} \, d\underline{x}\right)^{(k+2)/k}$$

$$= \frac{1}{\sigma_X^2} \left(\int f_1(x_1)^{k/k+2} dx_1 \ldots \int f_k(x_k)^{k/k+2} dx_k\right)^{(k+2)/k}$$

$$= \frac{1}{\sigma_X^2} \prod_{i=1}^{k} \left(\int f_i(x)^{k/k+2} dx\right)^{(k+2)/k}$$

(6) $X_1,\ldots,X_k$ independent and identical (IID) with variance σ^2

$$\beta_{X,k} = \frac{1}{\sigma^2} \left(\int f_1(x)^{k/k+2} dx\right)^{k+2}$$

(6') If $X_1,\ldots,X_k$ and $Y_1,\ldots,Y_k$ have the same marginal distributions, but $Y_1,\ldots,Y_k$ are independent, then

$\beta_{X,k} \le \beta_{Y,k}$ with equality iff the X_i's are independent.

(This illustrates how dependence among the X_i's (equivalently memory in the source) reduces the value of β_k.)

(7) Suppose $X_1,\ldots,X_k$ are Gaussian

(a) Independent case

$$\beta_{X,k} = 2\pi \left(\frac{k+2}{k}\right)^{(k+2)/2} \frac{\left(\prod_{i=1}^{k} \sigma_i^2\right)^{1/k}}{\sigma_X^2}$$

Derivation: From (5)

$$\beta_{X,k} = \frac{1}{\sigma_X^2} \prod_{i=1}^{k} \left(\int \left(\frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left\{ - \frac{x^2}{2\sigma_i^2} \right\} \right)^{k/(k+2)} dx \right)^{(k+2)/k}$$

$$= \frac{1}{\sigma_X^2} \prod_{i=1}^{k} \left(\int \frac{1}{\sqrt{2\pi\sigma_i^2 \frac{k+2}{k}}} \exp\left\{ - \frac{x^2}{2\sigma_i^2 \frac{k+2}{k}} \right\}^{k/(k+2)} \times \frac{\left(2\pi\sigma_i^2 \frac{k+2}{k}\right)^{1/2}}{\left(2\pi\sigma_i^2\right)^{k/(2(k+2))}} dx \right)^{(k+2)/k}$$

$$= \frac{1}{\sigma_X^2} \prod_{i=1}^{k} \left(\left(2\pi\sigma_i^2\right)^{1/(k+2)} \left(\frac{k+2}{k}\right)^{1/2} \right)^{(k+2)/k}$$

$$= 2\pi \left(\frac{k+2}{k}\right)^{(k+2)/2} \frac{\left(\prod_{i=1}^{k} \sigma_i^2\right)^{1/k}}{\sigma_X^2}$$

(b) IID Gaussian case

$$\beta_k = 2\pi \left(\frac{k+2}{k}\right)^{(k+2)/2} \to \beta_\infty = 2\pi e = 17.1 \text{ as } k \to \infty$$

Note: $\left(\frac{k+2}{k}\right)^{(k+2)/2} = \exp\left\{\frac{k+2}{2} \ln \frac{k+2}{k}\right\} = \exp\left\{\frac{k+2}{2} \ln\left(1+\frac{2}{k}\right)\right\}$

$$\cong \exp\left\{\frac{k+2}{2}\,\frac{2}{k}\right\} = \exp\left\{\frac{k+2}{k}\right\} \to e$$

(c) Correlated Gaussian random vector with covariance matrix K

$$\beta_k = 2\pi\left(\frac{k+2}{k}\right)^{(k+2)/2} \frac{|K|^{1/k}}{\sigma^2}$$

Derivation: We assume $E\,\underline{X} = \underline{0}$ because the mean does not affect β. We find β_k by transforming $\underline{X}$ to an independent vector $\underline{U}$ via an orthogonal transform. From (3), $\beta_k = \beta_{U,k}$; $\beta_{U,k}$ can be found from (a) above.

Accordingly, let $\underline{U} = A\,\underline{X}$, where A is the Karhunen-Loeve transform, i.e. its rows $\underline{z}_1,\ldots,\underline{z}_k$ are an orthonormal set of eigenvectors for K. Let $\lambda_1,\ldots,\lambda_k$ be the corresponding eigenvalues. Then

$$K_U = E\,\underline{U}\,\underline{U}^t = E\,A\underline{X}\,\underline{X}^t A^t = A\,E\,\underline{X}\,\underline{X}^t A^t = A\,K_X\,A^t$$

$$= A\,[\lambda_1 \underline{z}_1 \ \ldots \ \lambda_k \underline{z}_k] = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix}$$

from which we see that $\underline{U}$ is uncorrelated and, also, independent since it is Gaussian. Using the fact that A is orthonormal and (a) above, we have

$$\beta_{X,k} = \beta_{U,k} = 2\pi\left(\frac{k+2}{k}\right)^{(k+2)/2} \frac{\left(\prod_{i=1}^{k} \sigma_i^2\right)^{1/k}}{\sigma_X^2}$$

$$= 2\pi\left(\frac{k+2}{k}\right)^{(k+2)/2} \frac{|K|^{1/k}}{\sigma^2}$$

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(8) Uniform density

(a) Independent (each X_i uniform on some interval)

$$\beta_k = 12\,\frac{\left(\prod_{i=1}^{k} \sigma_i^2\right)^{1/k}}{\sigma^2}$$

(b) IID uniform on an interval

$$\beta_k = 12$$

Notice that it is the same for all k.

(c) Uniform on an arbitrary k-dimensional set B

$$\beta_k = \frac{\mathrm{vol}(B)^{2/k}}{\sigma^2}$$

(9) Laplacian density $\left(f_{\underline{X}}(x) = \frac{1}{\sqrt{2}} e^{-\sqrt{2}|x|},\ \sigma^2 = 1\right)$

(a) Independent

(b) IID

$$\beta_k = 2\left(\frac{k+2}{k}\right)^{k+2} \rightarrow \beta_\infty = 2e^2 = 14.8 \quad \text{as } k \to \infty$$

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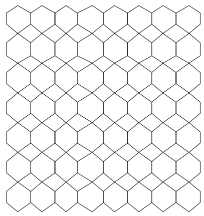
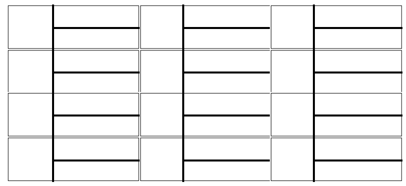
Might Gersho's Conjecture Be Wrong?

+ One plausible alternative is the best quantizers have, locally, a "periodic tesselation". In this case, we would take $m_k^*(x)$ to be the weighted average of the NMI's of one "period" (average weighted by volumes).





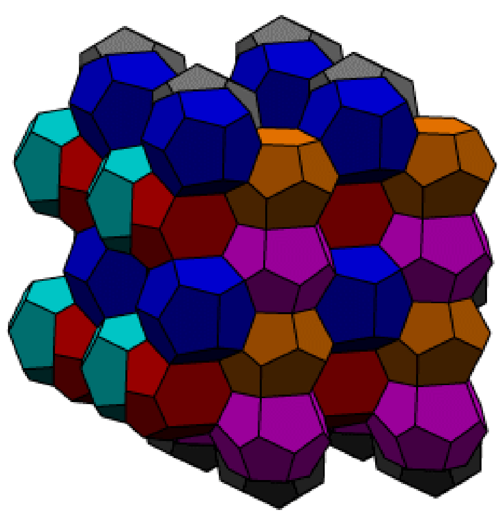
Bennett's integral does not apply to the first because the cells have differing volumes

Alternatives: pentagon and two diamonds, or 12 sided polygons and gaps filled with equilateral triangles,

+ Another possibility: the best quantizers have no periodicity or regularity. This seems unlikely.

The Weaire-Phelan Partition

It's a periodic tesselation/lattice with 3 types of cells. In particular the partition is formed by a tesselating a fundamental unit consisting of two pentagonal dodecahedra (12 faces, each is 5-sided) and six 14-hedra (2 hexagonal faces, 4 pentagonal faces of one kind and 6 pentagonal faces of another kind).



Average NMI of W-P partition with minimal NMI = 0.078735

NMI of truncated octahedron = 0.078543

- In any case, without using any conjectures and without using Bennett's integral, Zador (1963) showed that for a source whose density is uniform on a unit cube

$$M^{2/k}\delta(k,M) \to m_k^* \text{ as } M \to \infty$$

where m_k^* is some constant depending only on the dimension k.
He then showed, that for general source densities satisfying reasonable conditions,

$$M^{2/k}\delta(k,M) \to m_k^* \sigma^2 \beta_k \, .$$

- Zador also showed

$$m_k^* \geq m(\text{k-dimensional sphere})$$

Asymptotic Properties of Optimal Quantizers

Let $S_{\underline{x}}$ denote the cell containing $\underline{x}$.

- **Cell volume**

$$|S_{\underline{x}}| \cong \frac{1}{M\lambda_k^*(\underline{x})} = \frac{c}{M f_{\underline{X}}(\underline{x})^{k/(k+2)}}$$

Smaller where f is larger, which is not surprising.

- **Cell probability**

$$\Pr(S_{\underline{x}}) \cong f_{\underline{X}}(\underline{x})\,|S_{\underline{x}}| \cong f_{\underline{X}}(\underline{x}) \frac{c}{M} f_{\underline{X}}(\underline{x})^{-k/(k+2)}$$

$$= \frac{c}{M} f_{\underline{X}}(\underline{x})^{2/(k+2)}$$

Larger where f is larger

- **Cell distortion**

$$\frac{1}{k}\int_{S_{\underline{x}}} ||\underline{x}-Q(\underline{x}')||^2 f_{\underline{X}}(\underline{x}')\, d\underline{x}' \cong \frac{1}{k} f_{\underline{X}}(\underline{x}) \int_{S_{\underline{x}}} ||\underline{x}'-Q(\underline{x}')||^2\, d\underline{x}'$$

$$= \frac{1}{k} f_{\underline{X}}(\underline{x})\, k\, m_k^* \, |S_{\underline{x}}|^{(k+2)/k}$$

$$= f_{\underline{X}}(\underline{x})\, m_k^* \frac{c}{M} f_{\underline{X}}(\underline{x})^{(-k/k+2)\ (k+2)/k}$$

$$= m_k^* \frac{c}{M}$$

Same for all $\underline{x}$; i.e. all cells contribute the same to the distortion

- **Conditional cell distortion**

$$\frac{1}{k}\int_{S_{\underline{x}}} ||\underline{x}-Q(\underline{x}')||^2 f_{\underline{X}}(\underline{x}'|\underline{X}\in S_{\underline{x}})\, d\underline{x}' = \frac{1}{k}\int_{S_{\underline{x}}} ||\underline{x}-Q(\underline{x}')||^2 \frac{f_{\underline{X}}(\underline{x}')}{Pr(S_{\underline{x}})}\, d\underline{x}'$$

$$= \frac{1}{Pr(S_{\underline{x}})}\, m_k^* \frac{c}{M}$$

$$= \frac{M}{c} f_{\underline{X}}(\underline{x})^{-2/(k+2)}\ m_k^* \frac{c}{M}$$

Inversely proportional to cell probability. Smaller where f is larger.

The OPTA Function for Stationary Sources

For a stationary sources, we can use VQ's of any dimension and we seek the least possible distortion of any VQ with rate R or less and any dimension. That is, we seek the "ultimate" VQ OPTA function:

$$\delta(R) = \inf_{\text{VQ's with rate R or less}} D(VQ)$$

$$= \inf_k \delta(k,R)$$

Equivalently,

$$S(R) = \sup_{\text{VQ's with rate R or less}} SNR(VQ)$$

$$= \sup_k S(k,R)$$

There is no VQ that achieves $\delta(R)$ exactly and no value of k such that $\delta(k,R) = \delta(R)$. However, by definition of "inf", for any R and any small tolerance $\varepsilon > 0$, there is a VQ with rate R or less and distortion $D \le \delta(R) + \varepsilon$, and there is a value of k such that $\delta(k,R) \le \delta(R) + \varepsilon$. That is, one can come arbitrarily close to achieving the inf's.

An "inf" is like a "min" except it works in cases where "min" does not.
Example,

$$\min_{x\in(0,1)} x^2 \text{ does not exist}$$

Defn: $\inf_{x\in G} f(x)$ = largest number y such that $y \le f(x)$ for all $x \in G$

Example,

$$\inf_{x\in(0,1)} x^2 = 0$$

because $0 \le x^2$ for all $x \in (0,1)$ and there is no larger number y such that $y \le x^2$ for all $x \in (0,1)$

Defn: $\sup_{x\in G} f(x)$ = smallest number y such that $y \le f(x)$ for all $x \in G$

"inf" and "sup" are short for "infimum" and "supremum"

covered in Math 451

Properties of the OPTA functions of Stationary Sources

The OPTA's have a decreasing trend as k increases. However, it is not known if $\delta(k,R) \ge \delta(k+1,R)$ for all k. All that is known is:

Fact: The OPTA function $\delta(k,R)$ is subbadditive in k; i.e.

$$\delta(k,R) \le \frac{k}{k+l}\,\delta(k,R) + \frac{l}{k+l}\,\delta(l,R) \quad \text{for any } k,l$$

Proof: Let Q = product of Q_k and Q_l, which are k and l dim'l VQ's with rate R or less, and with $D(Q_k) \cong \delta(k,R)$ and $D(Q_l) \cong \delta(l,R)$. As in Problem 8, HW 2.

$$R(Q) = \frac{k}{k+l}\,R(Q_k) + \frac{l}{k+l}\,R(Q_l) \le R$$

$$D(Q) = \frac{k}{k+l}\,D(Q_k) + \frac{l}{k+l}\,D(Q_l) \cong \frac{k}{k+l}\,\delta(k,R) + \frac{l}{k+l}\,\delta(l,R)$$

Therefore,

$$\delta(k,R) \le D(Q) \cong \frac{k}{k+l}\,\delta(k,R) + \frac{l}{k+l}\,\delta(l,R)$$

Fact: Subadditivity implies

$$\delta(1,R) \ge \delta(k,R) \ge \delta(mk,R) \quad \text{for any } m,k$$

$$\delta(R) = \inf_k \delta(k,R) = \lim_{k\to\infty} \delta(k,R)$$

Proof: For any subadditive sequence, lim = inf (cf. Gallager, p. 112, 113)

Other Properties of the OPTA Function

For a stationary source with continuous random variables and fixed-rate VQ:

$\delta(0) = \sigma^2$ (recall that $\delta_{vq,k}(0) = 0$)

$\delta(R)$ decreases monotonically toward zero as $R \to \infty$.

$\delta(R)$ is a continuous, convex cup function of R.

This is not like $\delta(k,R)$, which has a stair-step form.

Sketch of proof of convexity: Given target rates R_1, R_2 and α, $0 < \alpha < 1$, we must show

$$\delta_{vq}(\alpha R_2+(1-\alpha)R_2) \leq \alpha\, \delta_{vq}(R_1) + (1-\alpha)\, \delta_{vq}(R_2).$$

First, suppose $\alpha = 1/2$. Let Q_1, Q_2 be VQ's with large dimension k, with rates at most R_1 and R_2 and distortions $D_1 \cong \delta(R_1)$ and $D_2 \cong \delta(R_2)$, respectively. (Q_1, Q_2 exist by the defn of the OPTA function.)

Consider another VQ, denoted Q, with dimension $2k$ created by using Q_1 followed by Q_2. (Q time shares between Q_1 and Q_2.) Then

$$R(Q) = \tfrac{1}{2}(R_1+R_2)$$

$$D(Q) = \tfrac{1}{2}(D_1+D_2).$$

Since $\delta_{vq}(\tfrac{1}{2}R_1+\tfrac{1}{2}R_2)$ = least dist'n of any quant, with rate $\tfrac{1}{2}R_1+\tfrac{1}{2}R_2$,

$$\delta_{vq}(\tfrac{1}{2}R_1+\tfrac{1}{2}R_2) \leq D(Q) = \tfrac{1}{2}(D_1+D_2) \cong \tfrac{1}{2}\big(\delta(R_1)+\delta(R_2)\big).$$

A somewhat sharper argument can demonstrate the above without the "$\cong$". It could also use a time sharing that applies to any value of α, thereby establishing convexity.

Proof of Continuity: Convexity implies continuity, except possibly at $R = 0$. But it can be shown that $\delta(R)$ is continuous at $R = 0$, too.

High Resolution Analysis of the OPTA's of Stationary Sources

Recall that for large R

$$\delta(k,R) \cong Z(k,R) \triangleq m_k^* \beta_k \sigma^2 2^{-2R}$$

What happens as k increases?

Recall: m_k^*'s have a decreasing trend, they are subadditive.

How about the β_k's? They, too, have a decreasing trend. It is not known if $\beta_{k+1} \le \beta_k$ for all k and all sources, however:

Fact: For a stationary source, β_k is submultiplicative; i.e.

$$\beta_{k+l} \le \left(\beta_k^k \beta_l^l\right)^{1/(k+l)} \quad \text{for any } k,l$$

equivalently $\log \beta_k$ is subbadditive:

$$\log \beta_{k+l} \le \frac{k}{k+l} \log \beta_k + \frac{l}{k+l} \log \beta \quad \text{for any } k,l$$

From submultiplicativity it follows that

$$\log \beta_1 \ge \log \beta_k \ge \log \beta_{mk} \quad \text{for every } m, k \ge 1$$

and also that $\lim_{k\to\infty} \log \beta_k = \inf_k \log \beta_k$

and consequently that

$$\beta_\infty \triangleq \lim_{k\to\infty} \beta_k = \inf_k \beta_k$$

Sketch of proof of Fact: Let $f_k(\underline{x})$ denote the k-dimensional density of $X_1,\ldots,X_k$. The proof uses the

Triple Holder inequality:

If f, g and h are nonnegative functions, and p, q, r are nonnegative numbers such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, then

$$\int f(\underline{x})\, g(\underline{x})\, h(\underline{x})\, d\underline{x} \le \left(\int f^p(\underline{x})\, d\underline{x}\right)^{1/p} \left(\int f^q(\underline{x})\, d\underline{x}\right)^{1/q} \left(\int f^r(\underline{x})\, d\underline{x}\right)^{1/r}$$

We apply this inequality to $\left(\sigma^2\beta_{k+l}\right)^{(k+l)/(k+l+2)} = \int f_{k+l}^{(k+l)/(k+l+2)}(\underline{x},\underline{y})\, d\underline{x}d\underline{y}$

with the integrand factored as $f_{k+l}^{(k+l)/(k+l+2)}(\underline{x},\underline{y}) = f(\underline{x},\underline{y})\, g(\underline{x},\underline{y})\, h(\underline{x},\underline{y})$, where

$$f(\underline{x},\underline{y}) = \left(f_{k+l}(\underline{x},\underline{y})\, f_k^{-2/(k+2)}(\underline{x})\right)^{k/(k+l+2)}, \quad g(\underline{x},\underline{y}) = \left(f_{k+l}(\underline{x},\underline{y})\, f_l^{-2/(l+2)}(\underline{y})\right)^{l/(k+l+2)}$$

$$h(\underline{x},\underline{y}) = \left(f_k^{k/(k+2)}(\underline{x})\, f_l^{l/(l+2)}(\underline{y})\right)^{2/(k+l+2)}$$

We let $p = (k+l+2)/k$, $q = (k+l+2)/l$ and $r = (k+l+2)/2$.

After simplifying, we find

$$\left(\sigma^2\beta_{k+l}\right)^{(k+l)/(k+l+2)} \le \left(\sigma^2\beta_k\right)^{k/(k+l+2)} \left(\sigma^2\beta_l\right)^{l/(k+l+2)}$$

which implies the desired result.

The decreasing trends for m_k^* and β_k indicate that one can't do better than to choose k large. Indeed, we need k to be large in order to approach the best possible performance. The following theorem summarizes.

Theorem: For a stationary source and large values of R,

$$\delta(R) \cong \frac{1}{2\pi e} \beta_\infty 2^{-2R} \triangleq Z(R)$$

Equivalently,

$$S(R) \cong 10 \log_{10} \frac{\sigma^2}{Z(R)} = 6.02 R - 10 \log_{10} \frac{1}{2\pi e} \beta_\infty$$

We again see 6 dB gain per bit.

Relationship of kth-order OPTA to overall OPTA

$$S(k,R) = 6.02 R - 10 \log_{10} m_k^* \beta_k$$

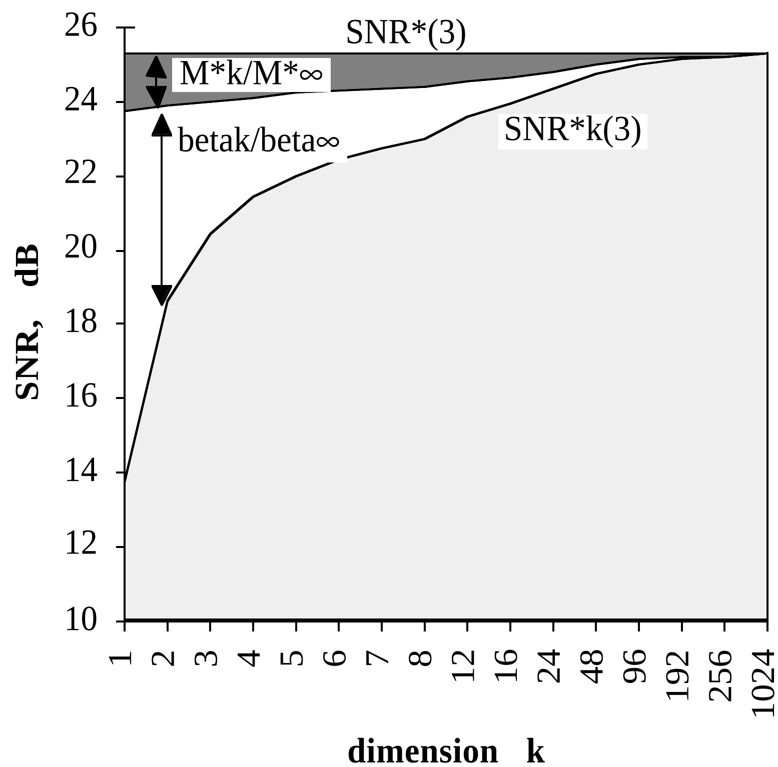
$$= 6.02 R - 10 \log_{10} m_\infty^* \beta_\infty - 10 \log_{10} \frac{m_k^*}{m_\infty^*} - 10 \log_{10} \frac{\beta_k}{\beta_\infty}$$

$$= S(R) - 10 \log_{10} \frac{m_k^*}{m_\infty^*} - 10 \log_{10} \frac{\beta_k}{\beta_\infty}$$

From this we see explicitly how $S_{vq,k}^*(R)$ improves with k through decreases in m_k^* and β_k.

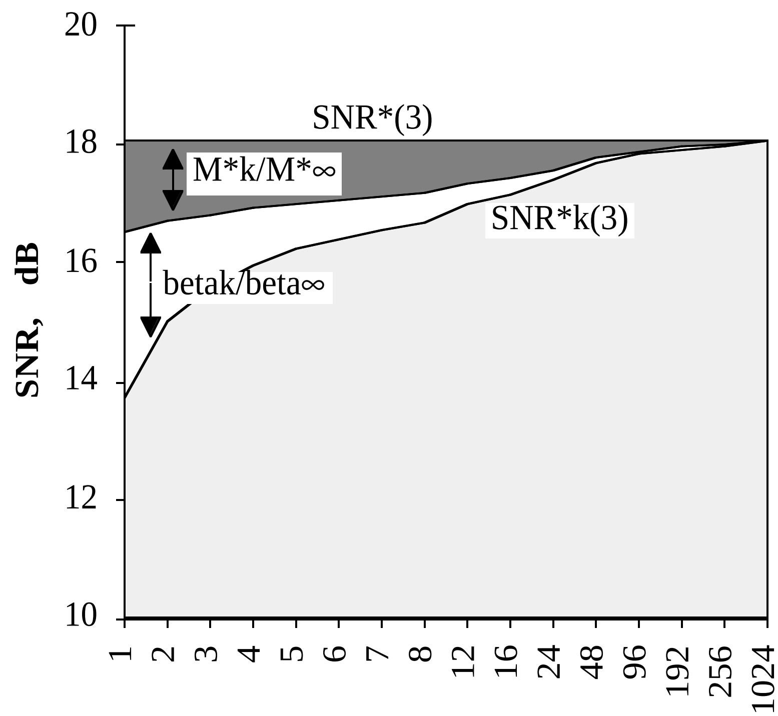
Example: Gauss-Markov Source, $\rho = .9$

Rate = 3



Example: IID Gaussian source

Rate: R = 3



Z-44

Question:

Why does β_k get better as k increases, even for an IID source?

We'll consider this question after discussing some properties of β_∞.

Z-45

Properties of β_∞

(1) For a Gaussian source $\beta_\infty = \dfrac{2\pi e Q}{\sigma^2}$

where $Q = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln S_X(\omega)\, d\omega\right\}$

= "one-step prediction error"

= MSE of optimum linear predictor for X_i based on $X_{i-1}, X_{i-2}, \ldots$

and $S_X(\omega)$ = power spectral density of X

Proof: This follows from the facts shown previously that

$$\beta_k = 2\pi \left(\frac{k+2}{k}\right)^{(k+2)/2} \frac{|K^{(k)}|^{1/k}}{\sigma^2}$$

$$2\pi \left(\frac{k+2}{k}\right)^{(k+2)/2} \to 2\pi e \text{ as } k \to \infty$$

and (in the discussion of transform coding to come later)

$$\lim_{k\to\infty} |K^{(k)}|^{1/k} = Q.$$

where $K^{(k)}$ is the k-dimen'l covariance matrix of the source.

(2) Upper bound

$$\beta_\infty \le \frac{2\pi e Q}{\sigma^2} \quad \text{with equality iff } X \text{ is Gaussian}$$

This shows that Gaussian sources have the largest $Z(R)$ among sources with a given power spectral density or autocorrelation function; i.e. they are hardest to quantize.

Note, for example, that even though an IID Laplacian source has a larger β_1 it has a smaller β_∞.

Proof: Postponed to variable-rate VQ discussion, for reasons that will be clear then.

(3) β_k tends to be smaller and to decrease more with k for sources with memory than for memoryless sources. (This is an admittedly rough rule of thumb.)

(4) For a stationary, Gaussian first-order autoregressive source with correlation coefficient ρ, $Q = 1-\rho^2$ and

$$\beta_k = 2\pi \left(\frac{k+2}{k}\right)^{(k+2)/2} (1-\rho^2)^{(k-1)/k} \to \beta_\infty = 2\pi e (1-\rho^2)$$

Why Fixed-Rate VQ Outperforms Fixed-Rate SQ

We want to understand what specific characteristics of vector quantization improve with dimension, and by how much.

We will first compare a fixed-rate k-dimensional VQ, denoted Q_k, to a fixed-rate scalar quantizer Q_1, both having rate R.

To make it fair, we compare characteristics (point density and inertial profile) of Q_k to those of the k-dimen'l "product" VQ, denoted $Q_{pr,k}$, formed by using Q_1 k times in succession.

The "product" codebook contains all k-tuples formed from scalar quant. levels.

In the "product" partition, each cell is the Cartesian product of the scalar cells corresponding to the components of its codevector.

The "product" quantization rule is

$$Q_{pr,k}(\underline{x}) = \big(Q_1(X_1), Q_2(X_2), \ldots, Q_k(X_k)\big)$$

It is easy to see that

$$R(Q_{pr,k}) = R(Q_1) = R$$

$$D(Q_{pr,k}) = D(Q_1) = \delta(1,R)$$

Assume the scalar quantizer has point density λ_1.

The cell $S_{\underline{x}}$ of the product quantizer containing $\underline{x}$ is a rectangle:

$$S_{\underline{x}} = \frac{1}{2^R \lambda_1(x_1)} \times \frac{1}{2^R \lambda_1(x_2)} \times \frac{1}{2^R \lambda_1(x_3)} \times \cdots \frac{1}{2^R \lambda_1(x_k)}$$

with volume

$$|S_{\underline{x}}| = \frac{1}{2^{kR}} \times \frac{1}{\lambda_1(x_1)\lambda_1(x_2)\ldots\lambda_1(x_k)}$$

The "product" point density is

$$\lambda_{pr,k}(\underline{x}) = \frac{1}{2^{kR}|S_{\underline{x}}|} = \lambda_1(x_1)\,\lambda_1(x_2)\,\ldots\,\lambda_k(x_k) \qquad \text{(It's a product!)}$$

The "product" inertial profile is

$$m_{pr,k}(\underline{x}) = \frac{1}{12}\,\frac{\frac{1}{k}\sum_{i=1}^{k}\frac{1}{\lambda_1(x_i)^2}}{\left(\prod_{i=1}^{k}\frac{1}{\lambda_1(x_i)^2}\right)^{1/k}}$$

Consider "loss" of the scalar quantizer (i.e. the product quantizer) relative to the VQ:

$$L = \frac{D(Q_1)}{D(Q_k)} = \frac{D(Q_{pr,k})}{D(Q_k)}$$

Applying Bennett's integral to both terms. Let

$$B(k, m, \lambda, f) = \int \frac{m(\underline{x})}{\lambda^{2/k}(\underline{x})} f(\underline{x})\, d\underline{x} = \text{Bennett's integral}$$

Then

$$L = \frac{D(Q_1)}{D(Q_k)} = \frac{D(Q_{pr,k})}{D(Q_k)} \cong \frac{B(k, m_{pr,k}, \lambda_{pr,k}, f)}{B(k, m_k, \lambda_k, f)}$$

$$= \frac{B(k, m_{pr,k}, \lambda_{pr,k}, f)}{B(k, m_k, \lambda_{pr,k}, f)} \times \frac{B(k, m_k, \lambda_{pr,k}, f)}{B(k, m_k, \lambda_k, f)}$$

$$= \text{cell shape loss} \times \text{point density loss}$$

$$= L_{ce} \times L_{pt}$$

Now assume k is large and Q_k is optimal, so that

$$m_k(\underline{x}) \cong m^*_\infty = \frac{1}{2\pi e} \quad \text{and} \quad \lambda_k(\underline{x}) \cong \lambda^*_k(\underline{x}) = c\, f^{k/(k+2)}(\underline{x})$$

Then loss of scalar quantizer relative to optimal high dim'l VQ is

$$L = \frac{1/12}{1/2\pi e} \times \frac{\int \frac{1}{k} \sum_{i=1}^{k} \frac{1}{\lambda_1^2(x_i)} f(\underline{x})\, d\underline{x}}{\int \frac{1}{\left(\prod_{i=1}^{k} \lambda_1^2(x_i)\right)^{1/k}} f(\underline{x})\, d\underline{x}} \times \frac{B(k, m^*_k, \lambda_{pr,k}, f)}{B(k, m^*_k, \lambda^*_k, f)}$$

$$= \text{space filling loss} \times \text{oblongitis loss} \times \text{point density loss}$$

$$= L_{sp} \times L_{ob} \times L_{pt}$$

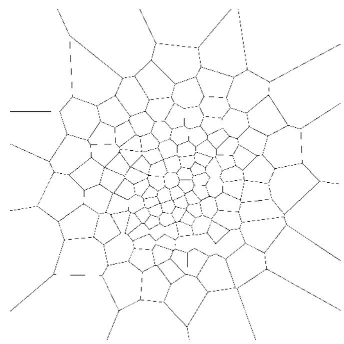
Now assume source is IID.

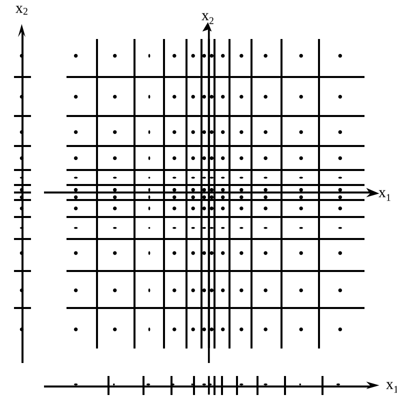
Consider the choice of $\lambda_1(x)$ to minimize the loss, i.e. to minimize $L_{ob} \times L_{pt}$.

- Choosing $\lambda_1(x)$ to be a constant on the set where $f_1(x)$ is large makes $L_{ob} \cong 1$. However, L_{pt} becomes very large.
- On the other hand choosing $\lambda_1(x) = f_1(x)$ causes

$$\lambda_{pr,k}(\underline{x}) \cong \prod_{i=1}^{k} f_1(x_i) = f(\underline{x}) \cong c\, f_k^{k/(k+2)}(\underline{x}) = \lambda^*_k(\underline{x}) \qquad (k \text{ large} \Rightarrow k/(k+2) \cong 1)$$

so that $L_{pt} = 1$. In other words scalar quantization can produce the optimal point density! This fact is often overlooked, because for IID Gaussian case, the product quantizer looks like it has a "cubical" point density, when it actually has a spherical one. Unfortunately, however, for this λ_1, However, $L_{ob} = \infty$.

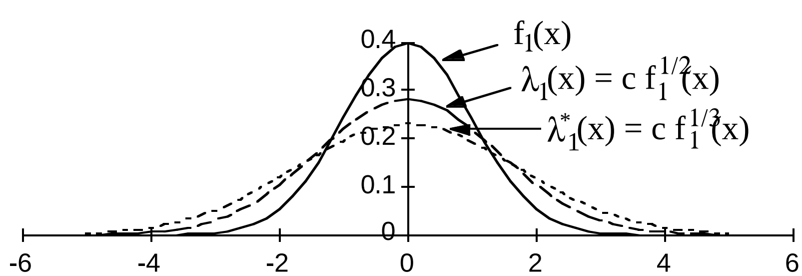




The point density that minimizes $L_{ob} \times L_{pt}$ is the compromise

$$\lambda_1(x) = c\, f_1(x)^{1/3}$$

This makes $\lambda_1(x)$ "flatter" (more uniform) in regions where $f_1(x)$ is large than the previous choice of $\lambda_1(x) = f(x)$. Therefore, the rectangular cells are more nearly cubical in the important region where $f_1(x)$ is large, so there is less oblongitis loss.



In summary, for an IID source the shortcomings of scalar quantization relative to high-dimensional VQ are

(a) The space filling loss $L_{sp} = 1.53$ dB, which is a measure of its inability to produce cells with smaller nmi than a cube.

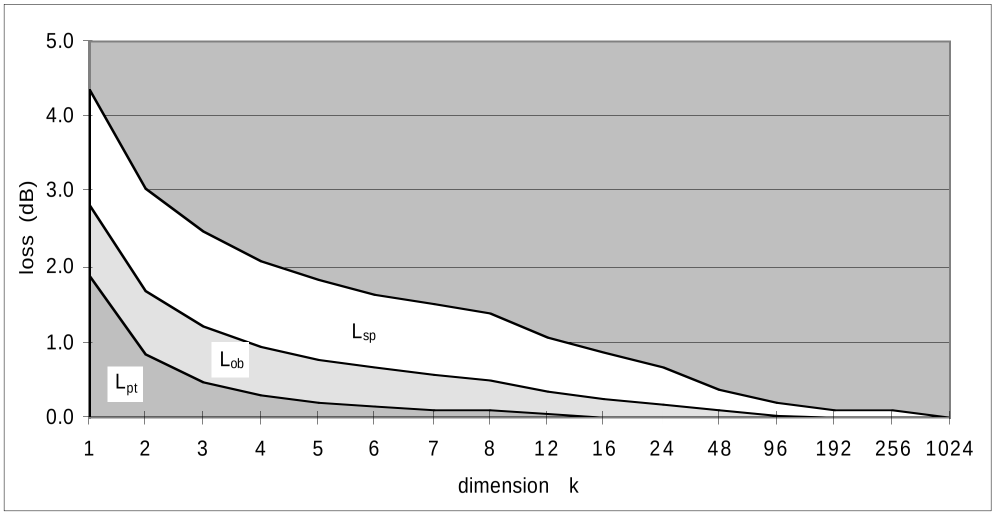
(b) The lack of sufficient degrees of freedom to simultaneously attain good inertial profile and good point density.

One can similarly decompose the loss of optimal k'-dimensional VQ ($k' \geq 2$) relative to high-dimensional VQ into space-filling, oblongitis and point density losses. As k' increases --

(a) The space filling loss decreases

(b) There are more degrees of freedom so that less compromise is needed between the k'-dimensional point density that minimizes oblongitis and that which minimizes point density loss. Consequently, when optimized, the oblongitis and point density losses are smaller.

Example: The losses for an IID Gaussian source



Memory Loss

For sources with memory, scalar and low-dimensional quantization suffers an additional loss, namely the inability to exploit or fully exploit the dependence between source samples. Both oblongitis and point density losses can be factored into two terms, one of which expresses the quantizers inability to fully exploit the source memory.