Conditional Codes

Example: X, stationary, alphabet: A = {1,2,3}, $P(X_n=i) = \frac{1}{3}$, $P(X_{n+1}=j|X_n=i) = \begin{cases} \frac{1}{2}, & j=i \\ \frac{1}{4}, & j \neq i \end{cases}$

first-order code

P(X=x)	x	codeword
1/3	1	0
1/3	2	10
1/3	3	11
rate		1.67

second-order code

$P(X_n=x_1,X_{n+1}=x_2)$	x_1x_2	codeword
1/6	11	000
1/12	12	0110
1/12	13	0111
1/12	21	100
1/6	22	001
1/12	23	101
1/12	31	110
1/12	32	111
1/6	33	010
avg length	19/6	
rate	1.583	

conditional code

	previous symbol X_{n-1}					
X_n	1		2		3	
1	1/2	0	1/4	11	1/4	10
2	1/4	10	1/2	0	1/4	11
3	1/4	11	1/4	10	1/2	0
$\bar{l}$		1.5		1.5		1.5

Rate = 1.5 bits/symbol = H_∞

Is code uniquely decodable?

Yes, knowing the previous symbol, the decoder knows which prefix code is being used to encode the current symbol.

It's a kind of "backward adaptive" coding.

Conditional Lossless Coding

Assume discrete-valued, stationary source X, as usual.

Basic idea:

When encoding X_n, use a code designed for the conditional probability distribution of X_n given the value of the previous symbol X_{n-1}. That is, when $X_{n-1}=a$, use a prefix code

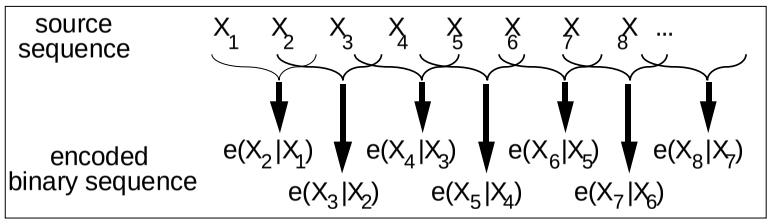
$C_a = \{\underline{v}_{a,1}, \underline{v}_{a,2}, \dots\}$ with lengths $\{l_{a,1}, l_{a,2}, \dots\}$

designed for

$p(1|a), p(2|a), p(3|a), \dots$

where

$p(i|a) = \Pr(X_n|X_{n-1}=a)$



Its conditional rate, given $X_{n-1}=a$, is

$$R_a = \bar{l}_a = \sum_i l_{a,i}\, p(i|a)$$

Its overall its rate is

$$R = \sum_a p(a)\, \bar{l}_a = \sum_a \sum_i l_{a,i}\, p(a)\, p(i|a)$$

If for each a, C_a is optimal for $(p(1|a), p(2|a), p(3|a), \ldots)$, then

$$H(X|a) \le R_a < H(X|a) + 1$$

where

$$H(X|a) = H(X_2|X_1=a) = -\sum_i p(i|a) \log_2 p(i|a) = \text{cond'l entropy of } X_2 \text{ given } X_1=a.$$

and the overall rate is

$$H(X_2|X_1) \le R < H(X_2|X_1) + 1$$

where

$$H(X_2|X_1) = \sum_a p(a)\, H(X_2|X_1=a) = -\sum_a \sum_i p(a)\, p(i|a) \log_2 p(i|a)$$

$$= \text{conditional entropy of } X_2 \text{ given } X_1$$

Note how similar this name is to that of $H(X_2|X_1=a)$.

Conditional Lossless Coding Theorem:

For a stationary source X,

$$H(X_2|X_1) \le R_c^* < H(X_2|X_1) + 1 ,$$

where R_c^* denotes the least rate of conditional codes used to encode source X.

Notes:

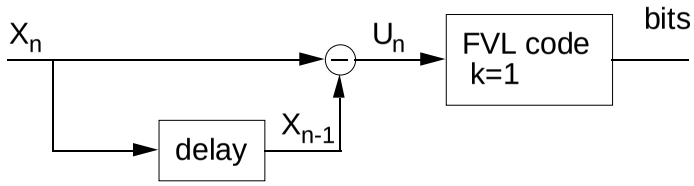
(1) This is a kind of backward adaptive coding. The encoder and decoder adapt the code based on the previous symbol.

(2) It is not absolutely essential that the C_a's be prefix codes. However, if not, one would have to require a kind of "mutual" unique decodability, namely, that any finite sequence of codewords from any of the codes be decodable in only one way.

(3) The upper bound $R_c^* < H(X_2|X_1)+1$ can be tightened by using the bound $R < H(X_2|X_1)+p_{c,max}+0.0836$, where $p_{c,max}$ is the largest value of $p(i|a)$ over all choices of i and a.

Predictive/Differential Coding

A special kind of conditional code



Example:

Alphabet of X: $A_X = \{0,1,2\}$.

Alphabet of U: $A_U = \{-2,-1,0,1,2\}$

Codebook for U: $C_U = \{110,01,00,10,111\}$

The equivalent conditional encoding table

X_n	previous symbol X_{n-1} 0	1	2
0	00	01	110
1	10	00	01
2	111	10	00

Complexity:

Compute $U_n = X_n - X_{n-1}$

Store just the one codebook C_U with $2|A|-1$ codewords

Rate:

$R \cong H(U)$

Often, $H(U) \cong H(X_2|X_1)$

Improvement:

Replace "delay" with a predictor

$\tilde{X}_n = g(X_{n-1}, X_{n-2}, \ldots)$

Widely used in lossless image coding, e.g. "lossless JPEG".

Improved Versions of Conditional Lossless Coding

There are two ways to improve conditional coding.

(1) mth-order conditional encoding: condition on m past symbols (rather than on one)

(2) (m,k) conditional block coding: conditionally encode k symbols at a time (rather than one).

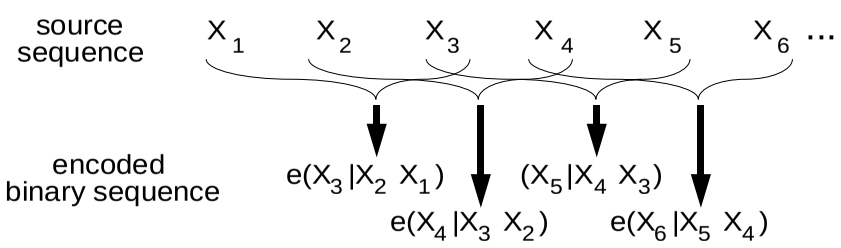
(1) mth-order conditional coding

In mth-order conditional coding, when encoding symbol X_n one uses a prefix codebook determined by the previous m symbols $X_{n-m},\ldots,X_{n-1}$.

That is, for every m-tuple $\underline{x} = (x_1,\ldots,x_m)$ in A^m, there is a prefix codebook

$C_{\underline{x}} = \{v_{\underline{x},1}, v_{\underline{x},2}, \ldots\}$ with lengths $\{l_{\underline{x},1}, l_{\underline{x},2}, \ldots\}$ and a distinct codeword for every symbol in A. When $(X_{n-m},\ldots,X_{n-1}) = \underline{x}$, then X_n is encoded with codebook $C_{\underline{x}}$.

The operation of a 2nd-order conditional code is illustrated below:



Given that $(X_{n-m},\ldots,X_{n-1}) = \underline{x}$, the average rate of the code is

$$R_{\underline{x}} = \bar{l}_{\underline{x}} = \sum_i l_{\underline{x},i}\, p(i|\underline{x})$$

where $p(i|\underline{x}) = \Pr(X_n=i|X_{n-m},\ldots,X_{n-1}=\underline{x})$, and the overall rate is

$$R = \sum_{\underline{x}} p(\underline{x})\, R_{\underline{x}} = \sum_{\underline{x}} p(\underline{x})\, \bar{l}_{\underline{x}}$$

where $p(\underline{x}) = \Pr(X_{n-m}^{n-1}=\underline{x}) = \Pr(X_1^m=\underline{x})$, (the latter by stationarity).

If for each $\underline{x}$, $C_{\underline{x}}$ is optimal for $(p(1|\underline{x}), p(2|\underline{x}), p(3|\underline{x}), \ldots)$, then

$$H(X_n|\underline{x}) \le R_{\underline{x}} < H(X_n|\underline{x}) + 1$$

where $H(X_n|\underline{x}) = H(X_n|X_{n-m},\ldots,X_{n-1}=\underline{x}) = -\sum_i p(i|\underline{x}) \log_2 p(i|\underline{x})$

= conditional entropy of X_2 given $X_{n-m},\ldots,X_{n-1}=\underline{x}$.

and the overall rate is $R = R^*_{1|m}$ where $R^*_{1|m}$ = least rate of any mth-order conditional code for the source X, and

$$H_{1|m} \le R^*_{1|m} < H_{1|m} + 1$$

where $H_{1|m} = H(X_n|X_{n-m},\ldots,X_{n-1}) = \sum_{\underline{a}} p(\underline{x})\, H(X_n|X_{n-m},\ldots,X_{n-1}=\underline{x})$

= conditional entropy of $H(X_n|X_{n-m},\ldots,X_{n-1})$

Again, note how similar this name is to that of $H(X_n|X_{n-m},\ldots,X_{n-1}=\underline{x})$.

Estimates of $H(X_n|X_{n-m},\ldots,X_{n-1})$ for various languages

$$H_{1|m} \triangleq H(X_n|X_{n-m},\ldots,X_{n-1})$$

	M=|A|	$\log_2 M$	$H(X_1)$	$H_{1\|2}$	$H_{1\|3}$	$H_{1\|4}$	$H_{1\|8}$	$H_{1\|12}$	$H_{1\|\infty}$
English	26	4.70	4.14	3.56	3.3				1.3
English	27	4.75	4.03	3.32	3.1				
English	26	4.70	4.12						
English	27	4.75	4.09	3.23	2.85	2.66	2.43	2.40	
French	26	4.70	3.98						
Spanish	26	4.70	4.02						
German	26	4.70	4.10						
Portuguese	26?	4.70?	3.92	3.51	3.15				
Russian	36	5.17	4.55	3.44	2.95	2.72	2.45	2.40	
Samoan	17	4.09	3.40	2.68	2.40	2.28	2.16	2.14	
Tamil	30	4.91	4.34						
Arabic	32	5.00	4.21	3.77	2.49				
Chinese	4700	12.20	9.63						

Reference: *Text Compression*, Bell, Cleary, Witten, p. 95.

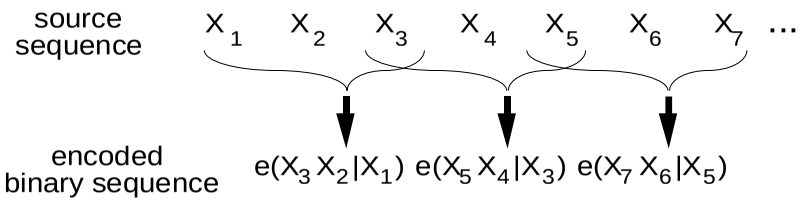
(2) Conditional Block Coding.

In (k|m)-conditional-block encoding, just as with FVL encoding with blocklength k, one divides the source sequence into blocks of length k.

Then as with mth-order conditional coding, when encoding a block beginning with symbol X_n, i.e. the block $(X_n,\ldots,X_{n+k-1})$ one uses a prefix codebook determined by the previous m symbols $X_{n-m},\ldots,X_{n-1}$.

That is, for every m-tuple $\underline{x} = (x_1,\ldots,x_m)$ in A^m, there is a prefix codebook $C_{\underline{x}} = \{\underline{v}_{\underline{x},1}, \underline{v}_{\underline{x},2}, \ldots\}$ with lengths $\{l_{\underline{x},1}, l_{\underline{x},2}, \ldots\}$ and one distinct codeword for every sequence in A^k. When $(X_{n-m},\ldots,X_{n-1}) = \underline{x}$, then $(X_n,\ldots,X_{n+k-1})$ is encoded with codebook $C_{\underline{x}}$.

The operation of a (1,2)-cond'l block code is illustrated below:



To allow more compact expressions, let X_u^v denote the sequence $(X_u,\ldots,X_v)$.

Given that $X_{n-m}^{n-1} = \underline{x}$, the average rate of the code is

$$R_{\underline{x}} = \tfrac{1}{k}\,\overline{l}_{\underline{x}} = \tfrac{1}{k}\sum_i l_{\underline{x},i}\, p(\underline{a}_i|\underline{x})$$

where $p(\underline{a}_i|\underline{x}) = \Pr(X_n^{n+k-1} = \underline{a}_i | X_{n-m}^{n-1} = \underline{x})$, and the overall rate is

$$R = \tfrac{1}{k}\sum_{\underline{x}} p(\underline{x})\, R_{\underline{x}} = \tfrac{1}{k}\sum_{\underline{x}} p(\underline{x})\, \overline{l}_{\underline{x}}$$

where $p(\underline{x}) = \Pr(X_{n-m}^{n-1} = \underline{x}) = \Pr(X_1^m = \underline{x})$, (the latter by stationarity).

If $C_{\underline{x}}$ is optimal for $(p(\underline{a}_1|\underline{x}), p(\underline{a}_2|\underline{x}), p(\underline{a}_3|\underline{x}), \ldots)$, then

$$\tfrac{1}{k}\, H(X_n^{n+k-1}|\underline{x}) \le R_{\underline{x}} < \tfrac{1}{k}\, H(X_n^{n+k-1}|\underline{x}) + \tfrac{1}{k}$$

where

$$H(X_n^{n+k-1}|\underline{x}) = H(X_n^{n+k-1}|X_{n-m}^{n-1} = \underline{x}) = -\sum_i p(\underline{a}_i|\underline{x}) \log_2 p(\underline{a}_i|\underline{x})$$

= conditional entropy of X^{n+k-1} given $X_{n-m}^{n-1} = \underline{x}$.

and the overall rate is

$$H_{k|m} \le R^*_{k|m} < H_{k|m} + \tfrac{1}{k}$$

where $R^*_{k|m}$ denotes the least rate of any (m,k) conditional block code for the source X,

$$H_{k|m} = \frac{1}{k} H(X_n^{n+k-1} | X_{n-m}^{n-1}) = \text{(k|m)-th order entropy}$$

and

$$H(X_n^{n+k-1} | X_{n-m}^{n-1}) = \sum_{\underline{x}} p(\underline{x})\, H(X_n^{n+k-1} | \underline{x}) = -\sum_{\underline{x}} \sum_i p(\underline{x})\, p(\underline{a}_i | \underline{x}) \log_2 p(\underline{a}_i | \underline{x})$$

$$= \text{conditional entropy of } X^{n+k-1} \text{ given } X_{n-m}^{n-1}$$

Again, note how similar this name is to that of $H(X_n^{n+k-1} | X_{n-m}^{n-1} = \underline{x})$.

We summarize and take the limit in the following.

Coding Theorem for Conditional Block Codes:

For a stationary source X and positive integers k, m

$$H_{k|m} \le R^*_{k|m} < H_{k|m} + \frac{1}{k},$$

where $R^*_{k|m}$ denotes the least rate of any (k|m) conditional block code encoding source X, and $H_{k|m} = \frac{1}{k} H(X_n^{n+k-1} | X_{n-m}^{n-1})$ is the (k|m)-th order entropy. Moreover,

$$\lim_{k\to\infty} R^*_{k|m} = H_\infty, \quad \text{for any } m$$

$$H_\infty \le \lim_{m\to\infty} R^*_{k|m} < H_\infty + \frac{1}{k}, \quad \text{for any } k.$$

Proof: The first statement was proved above. The second and third statement follow from the first and Properties 15 and 16, given later, which show that

$$\lim_{k\to\infty} H_{k|m} = H_\infty \text{ for any m, and } \lim_{m\to\infty} H_{k|m} = H_\infty \text{ for any k}$$

This theorem shows that we can approach the least possible rate, H_∞, in a number of ways.

This theorem also applies to (unconditional) FVL block codes, by considering FVL block codes to be (k|m) conditional block codes with $m = 0$ and by defining $H_{k|0} = H_k$. In this case, the theorem includes the coding theorem for FVL block codes.

Complexity of conditional and conditional block codes

Suppose the source alphabet contains M symbols.

A direct implementation of an (k|m) conditional block code requires storing the M^k codewords of each of the M^m codes $C_{\underline{x}}$. That is, the total storage required for encoding or decoding is proportional to M^{k+m}.

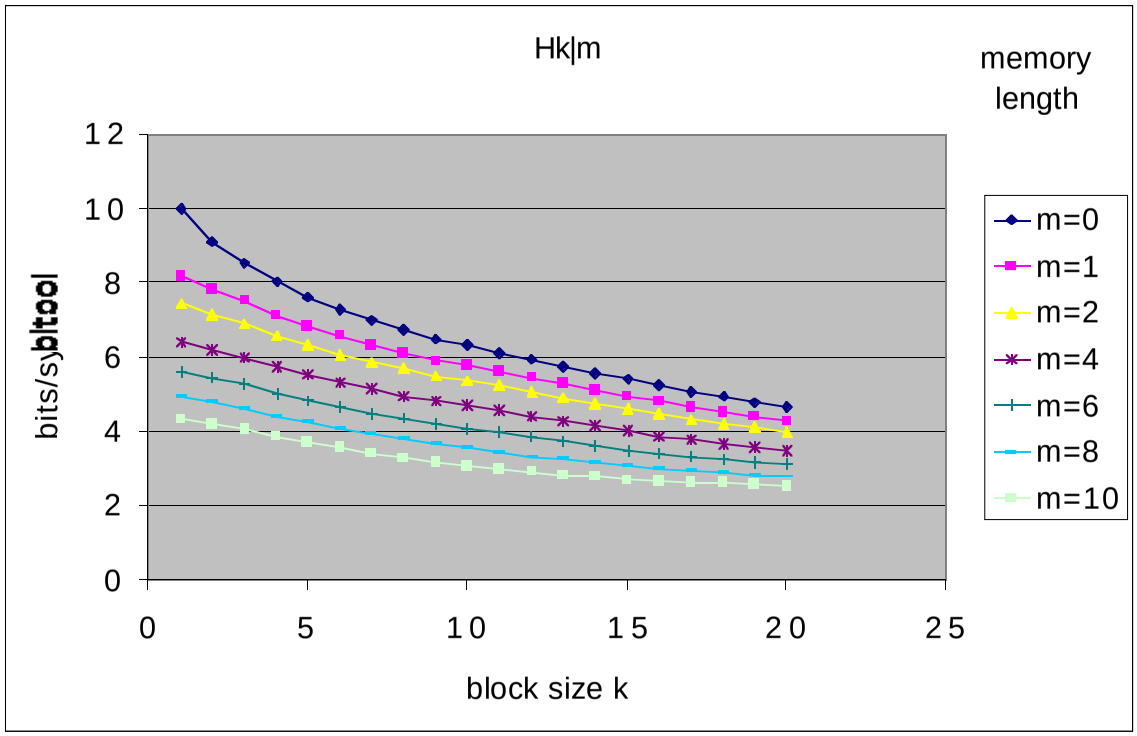
Thus complexity grows exponentially with m and k.

Suppose we fix a value K and require that $k+m = K$, thereby fixing the complexity, at least approximately. What values of k and m lead to the lowest rate?

The answer depends on the source, so there is no universally best choice. One can use $H_{k|m} + 1/k$ as a guide to the choice of k and m.

The facts given later show that if $k+m=K$, $H_{1|K-1}$ is smallest.

Example: A hypothetical 20th-order Markov source with H_1= 10, H_∞ = 2



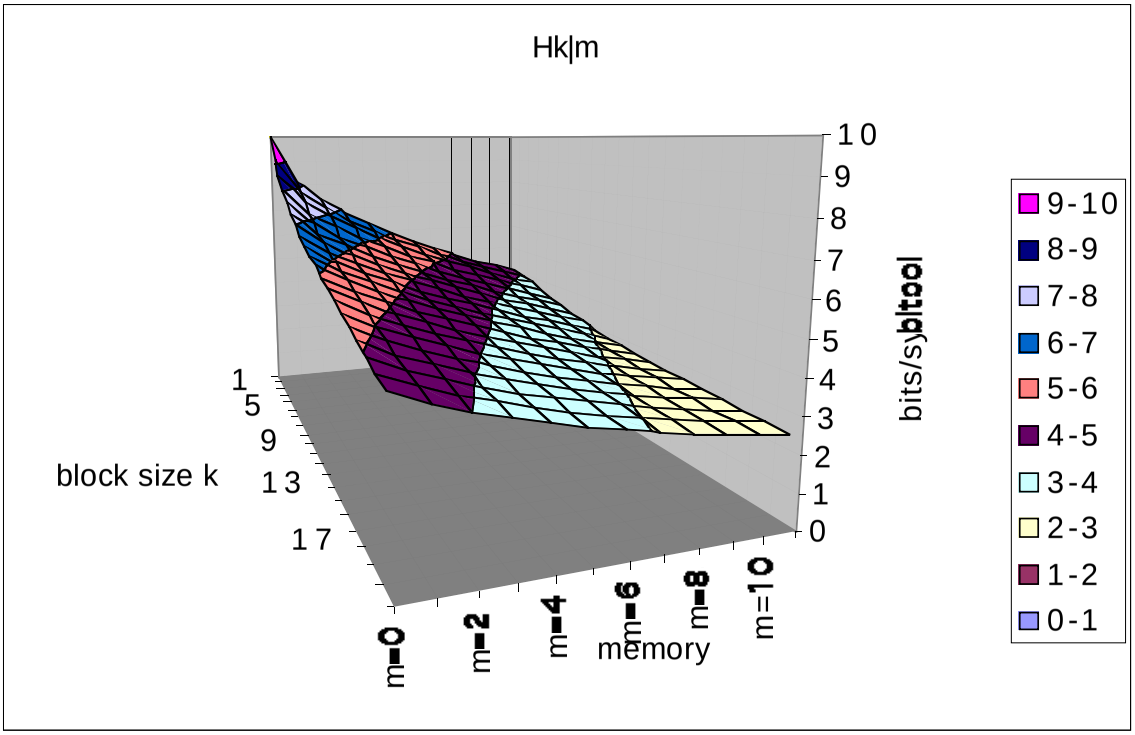


Table of $H_{k|m}$

k =	m = 0	1	2	3	4	5	6	7	8	9	10	11	12
1	10	8.21	7.47	6.9	6.42	**6**	5.62	5.27	4.94	**4.6**	4.34	4.07	3.8
2	9.11	7.84	7.19	6.66	**6.2**	5.81	5.44	5.1	**4.8**	4.49	4.21	3.94	3.68
3	8.56	7.53	6.93	**6.4**	6.01	5.63	5.28	**4.9**	4.64	4.35	4.07	3.81	**3.6**
4	8.03	7.15	**6.6**	6.15	5.75	5.38	**5**	4.73	4.43	4.15	3.88	**3.6**	3.38
5	7.62	**6.8**	6.34	5.91	5.52	**5.2**	4.85	4.54	4.25	3.98	**3.7**	3.47	3.23
6	**7.3**	6.58	6.1	5.69	**5.3**	4.99	4.67	4.38	4.1	**3.8**	3.57	3.33	3.09
7	7	6.34	5.89	**5.5**	5.15	4.82	4.51	4.22	**3.9**	3.69	3.44	3.2	2.96
8	6.74	6.13	**5.7**	5.32	4.98	4.66	4.36	**4.1**	3.81	3.56	3.31	3.07	2.84
9	6.51	**5.9**	5.52	5.15	4.82	4.51	**4.2**	3.94	3.68	3.43	3.19	2.95	2.75
10	**6.3**	5.77	5.37	5.02	4.69	**4.4**	4.1	3.83	3.57	3.33	3.09	2.86	2.67
11	6.14	5.62	5.23	4.88	**4.6**	4.27	3.99	3.72	3.47	3.22	2.99	2.78	2.61
12	5.95	5.45	5.07	**4.7**	4.42	4.13	3.86	3.6	3.35	3.12	2.91	2.71	2.56
13	5.76	5.28	**4.9**	4.59	4.28	4	3.73	3.47	3.24	3.04	2.84	2.66	2.52
14	5.59	**5.1**	4.77	4.45	4.15	3.87	3.61	3.37	3.15	2.96	2.78	2.61	2.48
15	**5.4**	4.97	4.63	4.31	4.02	3.75	3.5	3.28	3.08	2.9	2.73	2.57	2.45
16	5.26	4.82	4.49	4.18	3.89	3.64	3.41	3.2	3.01	2.84	2.68	2.54	2.42
17	5.1	4.68	4.35	4.05	3.78	3.54	3.32	3.13	2.95	2.79	2.64	2.5	2.4
18	4.95	4.54	4.22	3.94	3.68	3.46	3.25	3.06	2.9	2.75	2.6	2.48	2.37
19	4.81	4.41	4.1	3.84	3.6	3.38	3.18	3.01	2.85	2.71	2.57	2.45	2.35
20	4.67	4.29	4	3.74	3.52	3.31	3.12	2.96	2.81	2.67	2.54	2.43	2.34

The values on a diagonal, such as those printed in bold, have the same complexity.

More Definitions and Properties of Entropy

For Two Random Variables

Definitions:

(a) The entropy of a pair of random variables X and Y (sometimes called their "joint" entropy) is

$$H(X,Y) = -\sum_{x,y} p(x,y) \log_2 p(x,y), \qquad \text{where } p(x,y) \triangleq \Pr(X=x \text{ and } Y=y).$$

Think of (X,Y) as <u>one</u> random variable with an alphabet consisting of pairs. Then the above is just the usual definition of entropy, and has the usual properties.

(b) The conditional entropy of random variable X given a particular value of random variable Y

$$H(X|Y=y) = -\sum_{x} p(x|y) \log_2 p(x|y), \qquad \text{where } p(x|y) = \Pr(X=x|Y=y).$$

This is an ordinary entropy -- namely the entropy of X when a specific value of Y is given -- and as such it has all the usual properties of entropy.

(c) Conditional entropy of random variable X given a random variable Y

$$H(X|Y) = \sum_{y} p(y)\, H(X|Y=y) = -\sum_{x,y} p(x,y) \log_2 p(x|y)$$

This is an average of the previous kind of conditional entropy.

Properties: (elementary in information theory)

1. $H(X|Y) \geq 0$ with equality iff X is a function of Y

   Proof: $H(X|Y=y) \geq 0$ for all y because it is an ordinary entropy. Since H(X|Y) is the average of such, it too is nonnegative. $H(X|Y) = 0$ if and only if $H(X|Y=y)=0$ for each y with positive probability. This happens if and only for each y with positive probability, there is a value x such that $\Pr(X=x|Y=y)=1$. In other words, Y determines X; i.e. X is a function of Y.

2. $H(X|Y) \leq H(X)$ with equality iff X and Y are independent.

   Proof: We'll show $H(X) - H(X|Y) \geq 0$ with equality iff X indep of Y.

$$\begin{aligned} H(X) - H(X|Y) &= -\sum_{x,y} p(x,y) \log_2 p(x) + \sum_{x,y} p(x,y) \log_2 \frac{p(x,y)}{p(y)} \\ &= -\sum_{x,y} p(x,y) \ln \frac{p(x)\,p(y)}{p(x,y)} \frac{1}{\ln 2} \\ &\geq -\sum_{x,y} p(x,y) \left(\frac{p(x)\,p(y)}{p(x,y)} - 1 \right) \frac{1}{\ln 2} \qquad \text{using } \ln z \leq z-1 \\ &= -\sum_{x,y} p(x)\,p(y) \frac{1}{\ln 2} + \sum_{x,y} p(x,y) \frac{1}{\ln 2} = 0. \end{aligned}$$

   Equality holds if and only if $p(x)\,p(y) = p(x,y)$ for all x,y; i.e. if and only if X and Y are independent.

3. Chain rule: $H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$

Proof:

$$H(X,Y) = -\sum_{x,y} p(x,y) \log_2 p(x)\, p(y|x)$$

$$= -\sum_{x,y} p(x,y) \log_2 p(x) \; -\sum_{x,y} p(x,y) \log_2 p(y|x)$$

$$= H(X) + H(Y|X).$$

Interchanging X and Y in this argument shows $H(X,Y) = H(Y)+H(X|Y)$.

4. $H(X,Y) \leq H(X) + H(Y)$
with equality iff X and Y are independent

Proof: Using Facts 3 and 2,

$$H(X,Y) = H(Y) + H(X|Y) \leq H(Y) + H(X)$$

with equality iff X and Y are independent.

5. $H(X) \leq H(X,Y)$ with equality iff Y is a function of X

Proof: By Fact 3, $H(X) = H(X,Y) - H(Y|X) \leq H(X,Y)$, where the inequality is from Fact 1, and equality holds iff $H(X|Y) = 0$, which by Fact 1 happens iff X is a function of Y.

More Than Two Random Variables

Definitions:

(d) Entropy of $X_1,\ldots,X_k$ (sometimes called their "joint entropy")

$$H(X_1,...,X_k) = -\sum_{\underline{x}} p(\underline{x}) \log_2 p(\underline{x}) \quad \text{where } \underline{x} = (x_1,\ldots,x_k)$$

(e) Conditional entropy of random variables $X_1,\ldots,X_k$ given particular values of random variables $Y_1,\ldots,Y_m$

$$H(X_1,\ldots,X_k|Y_1,\ldots,Y_m=y_1,\ldots,y_m) = -\sum_{\underline{x}} p(\underline{x}|\underline{y}) \log_2 p(\underline{x}|\underline{y}), \quad \text{where } \underline{y} = (y_1,\ldots,y_m)$$

(f) Conditional entropy of random variables $X_1,\ldots,X_k$ given random variables $Y_1,\ldots,Y_m$

$$H(X_1,\ldots,X_k|Y_1,\ldots,Y_m) = \sum_{\underline{y}} p(\underline{y})\, H(X_1,\ldots,X_k|Y_1,\ldots,Y_m=y_1,\ldots,y_m)$$

$$= -\sum_{\underline{x},\underline{y}} p(\underline{x},\underline{y}) \log_2 p(\underline{x}|\underline{y})$$

Note: If one thinks of $(X_1,\ldots,X_k)$ and $(Y_1,\ldots,Y_m)$ each as a single random variable with a vector-valued alphabet, then the above formulas are the same as the corresponding "two-random variable" formulas.

Properties:

6. $H(Y_1,...,Y_n|X_1,\ldots,X_m) \le H(Y_1,\ldots,Y_n|X_1,\ldots,X_{m'})$, $0 \le m' < m$,

   with equality iff $Y_1,\ldots,Y_n$ is conditionally independent of $X_{m'+1},\ldots,X_m$ given $X_1,\ldots,X_{m'}$.

   Proof: Like that of Fact 2.

7. Chain rule:

   $$H(X_1,...,X_k) = H(X_1) + H(X_2|X_1) + H(X_3|X_1,X_2)$$
   $$+ H(X_4|X_1,X_2,X_3) + ... + H(X_k|X_1,X_2,\ldots,X_{k-1})$$

   Proof: Like that of Fact 3.

8. $H(X_1,...,X_k) \le H(X_1) + H(X_2) + ... + H(X_k)$

   with equality iff the X_i's are independent

   Proof: Follows from the chain rule and the fact that $H(X_i|X_1,\ldots,X_{i-1}) \le H(X_i)$ with equality if and only if X_i is independent of $X_1,\ldots,X_{i-1}$.

For Stationary Random Processes

Let $\{X_k\}$ be stationary random process.

Definitions:

(g) $H_k \triangleq \frac{1}{k} H(X_1,...,X_k)$

(h) $H_{1|m} \triangleq H(X_n|X_{n-m},X_2,\ldots,X_{n-1})$ $\quad (H_{1|0} \triangleq H(X_1) = H_1)$

(i) $H_{k|m} \triangleq \frac{1}{k} H(X_n^{n+k-1}|X_{n-m}^{n-1})$ $\quad (H_{1|m} = H_{1|m})$

(j) $H_\infty \triangleq \lim_{k\to\infty} H_k$ = entropy rate of X

Properties:

9. $H_{1|m+1} \leq H_{1|m}$

   Proof: Follows from Property 6 and stationarity.

10. $H_k = \frac{1}{k}\left(H_1 + H_{1|1} + H_{1|2} + \ldots + H_{1|k-1}\right) \geq H_{1|k-1} \geq H_{1|k}$

   Proof: $H_k = \frac{1}{k} H(X_1,\ldots,X_k)$

   $= \frac{1}{k}\left(H(X_1) + H(X_2|X_1) + H(X_3|X_1,X_2) + \ldots + H(X_k|X_1,X_2,\ldots,X_{k-1})\right)$
   by chain rule

   $= \frac{1}{k}\left(H_1 + H_{1|1} + H_{1|2} + \ldots + H_{1|k-1}\right)$ by stationarity

   $\geq \frac{1}{k}\left(H_{1|k-1} + H_{1|k-1} + H_{1|k-1} + \ldots + H_{1|k-1}\right)$ by Prop. 9.

   $= H_{1|k-1} \geq H_{1|k}$ by Prop. 9

11. $H_{k+1} \leq H_k$

   Since H_k's are nonincreasing and bounded below by zero, they must have a limit. Hence, H_∞ is a well-defined quantity.

   Proof: By Prop. 10, H_k is the average of k terms $H_1, H_{1|1}, H_{1|2}, \ldots, H_{1|k-1}$.

   Similarly, H_{k+1} is average of k+1 terms $H(X_1), H_{1|1}, H_{1|2}, \ldots, H_{1|k-1}, H_{1|k}$.

   Since the extra term in H_{k+1} is no larger than all other terms, $H_{k+1} \leq H_k$.

12. $\lim_{k\to\infty} H_{1|k} = \lim_{k\to\infty} H_k \stackrel{\Delta}{=} H_\infty$

   Proof: Since the $H_{1|k}$'s are nonincreasing with k and bounded below by zero, they must have a limit. Since by Prop. 10, H_k is the average of the k terms $H(X_1), H_{1|1}, H_{1|2}, \ldots, H_{1|k-1}$, the limit of the $H_{1|k}$'s equals the limit of the H_k's, which by definition is H_∞.

13. $H_{k|m} = \frac{1}{k}\left(H_{1|m} + H_{1|m+2} + \ldots + H_{1|k+m}\right)$

   Proof: $H_{k|m} = \frac{1}{k} H(X_n^{n+k-1}|X_{n-m}^{n-1})$

   $= \frac{1}{k}\left(H(X_n|X_{n-m}^{n-1}) + H(X_{n+1}|X_{n-m}^{n}) + \ldots + H(X_{n+k-1}|X_{n-m}^{n+k-2})\right)$

   $= \frac{1}{k}\left(H_{1|m} + H_{1|m+2} + \ldots + H_{1|k+m}\right)$ by stationarity

14. $H_{1|k+m} \leq H_{k|m} \leq H_k$

   Proof: (a) $H_{k|m} = \frac{1}{k} H(X_n^{n+k-1}|X_{n-m}^{n-1}) \leq \frac{1}{k} H(X_n^{n+k-1}) = H_k$

   (b) By Prop. 13, $H_{k|m}$ is the average of terms, each of which is $\geq H_{1|k+m}$. Therefore, $H_{k|m} \geq H_{1|k+m}$.

15. $\lim_{k\to\infty} H_{k|m} = H_\infty$ for any m

Proof: Recall from Prop. 13 that $H_{k|m} = \frac{1}{k}\left(H_{1|m} + H_{1|m+2} + \ldots + H_{1|k+m}\right)$

It follows that the limit as $k\to\infty$ of $H_{k|m}$ equals $\lim_{k\to\infty} H_{1|k+m} = H_\infty$

16. $\lim_{m\to\infty} H_{k|m} = H_\infty$ for any k

Proof: Recall from Prop. 13 that $H_{k|m} = \frac{1}{k}\left(H_{1|m} + H_{1|m+2} + \ldots + H_{1|k+m}\right)$

It follows that the limit as $m\to\infty$ of $H_{k|m}$ equals $\lim_{m\to\infty} H_{1|k+m} = H_\infty$

Examples:

A. IID source:

$H(X_1) = H_1 = H_k = H_\infty = H_{1|1} = H_{1|k} = H_{k|m}$, for all k,m

B. (first-order) Markov source:

Definition: A source is first-order Markov if

$p(x_m|x_1,\ldots,x_{m-1}) = p(x_m|x_{m-1})$ for all m and $x_1,\ldots,x_m$

Then for any $m \geq 1$,

$$H_{1|m} = H(X_m|X_1,\ldots,X_{m-1}) = -\sum_{x_1\ldots x_m} p(x_1,\ldots,x_m)\log_2 p(x_m|x_1,\ldots,x_{m-1})$$

$$= -\sum_{x_1\ldots x_m} p(x_1,\ldots,x_m)\log_2 p(x_m|x_{m-1}) = -\sum_{x_{m-1},x_m} p(x_{m-1},x_m)\log_2 p(x_m|x_{m-1})$$

$$= H(X_m|X_{m-1}) = H(X_2|X_1) = H_{1|1} \quad \text{by stationarity}$$

Since all the $H_{1|m}$'s equal $H_{1|1}$,

$$H_\infty = \lim_{m\to\infty} H_{1|m} = H_{1|1}$$

By Prop. 10 and the above

$$H_k = \frac{1}{k}(H_1+H_{1|1}+H_{1|2}+\ldots+H_{1|k-1}) = \frac{1}{k}(H_1+(k-1)H_{1|1}) = \frac{1}{k}H_1+\frac{k-1}{k}H_{1|1}$$

from which we see that H_k converges down to H_∞, as it should.

Similarly, one can show

$$H_{k|m} = H_{1|1} = H_\infty \quad \text{for all } k,m$$

Because $H_{1|1} = H_\infty$, we see that first-order conditional coding can come within one bit of optimal for first-order Markov sources. In other words there is no reason to use 2nd or higher order conditional coding, and the only reason to use block coding is to reduce the possibly one bit of redundancy.

C. nth-order Markov source:

Definition: A source is nth-order Markov if

$$p(x_k|x_1,\ldots,x_{k-1}) = p(x_k|x_{k-n},\ldots,x_{k-1}) \quad \text{for all } k>n \;\&\; x_1,\ldots,x_k$$

Then as in the case of 1st-order Markov sources, it can be shown that if $m \geq n$,

$$H_{1|m} = H_{k|m} = H_{1|n} = H_\infty, \quad \text{for every } k,$$

and that if $k \geq n$,

$$H_k = \frac{1}{k}(H_1 + H_{1|1} + H_{1|2} + \ldots + H_{1|n-1} + (k-n)H_{1|n})$$

which converges down to $H_\infty = H_{1|n}$.

Because $H_{1|n} = H_\infty$, we see that nth-order conditional coding can come within one bit of optimal for nth-order Markov sources. In other words there is no reason to use higher order conditional coding, and the only reason to use block coding is to reduce the possibly one bit redundancy.