Recall: $D \cong \frac{1}{12}\left(\prod_{j=1}^{k}\sigma_j^2\alpha_j\right)^{1/k} 2^{-2R}$. For Gaussian case, α_j is same all j.

With the KLT, $\sigma_1,\dots,\sigma_k = \lambda_1,\dots,\lambda_k$ and $\Gamma = \left(\prod_{j=1}^{k}\sigma_k^2\right)^{1/k} = \left(\prod_{j=1}^{k}\lambda_j\right)^{1/k} = |K_{\underline{X}}|^{1/k}$.

Therefore.

The OPTA Function for k-dimensional Transform Coding applied to a Stationary, Gaussian Source:

For large R,

$$\delta_{tr}(k,R) \cong \frac{1}{12}|K_{\underline{X}}|^{1/k}\alpha_{G,1}\, 2^{-2R},$$

where

$$\alpha_{G,1} = \begin{cases} 2\pi 3^{3/2} = 32.6 & \text{for FRC} \\ 2\pi e = 17.08 & \text{for VRC} \end{cases}$$

The best transform is the KLT, i.e. rows are orthonormal eigenvectors for $K_{\underline{X}}$.
The resulting coefficients $U_1,\dots,U_k$ are uncorrelated (indeed, independent),
Their variances $\sigma_1^2,\dots,\sigma_k^2$ equal the eigenvalues $\lambda_1,\dots,\lambda_k$ of $K_{\underline{X}}$.
The rate allocated to the ith coefficient is: $R_i = R + \frac{1}{2}\log_2 \frac{\lambda_i}{|K_{\underline{X}}|^{1/k}}$
The resulting coeff. distortions are all the same and equal to $\delta_{tr}(k,R)$.

COMPARISONS (GAUSSIAN CASE)

Optimal k-Dimensional Transform coding

$$\delta_{tr}(k,R) \cong \frac{1}{12}|K_{\underline{X}}|^{1/k}\alpha_{G,1}\, 2^{-2R}, \qquad \alpha_{G,1} = \begin{cases} 2\pi 3^{3/2} = 32.6 & \text{for FRC} \\ 2\pi e = 17.08 & \text{for VRC} \end{cases}$$

Optimal Scalar Quantization

$$\delta_{sq}(R) \cong \frac{1}{12}\sigma^2\alpha_{G,1}\, 2^{-2R}$$

SNR Gain over scalar quantization.

$$10\log_{10}\frac{\delta_{sq}(R)}{\delta_{tr}(k,R)} \cong 10\log_{10}\frac{\sigma_X^2}{|K_{\underline{X}}|^{1/k}}$$

Optimal k-dimensional VQ: For a stationary, Gaussian source

$$\delta_{vq}(k,R) \cong m_k^*\sigma_X^2\alpha_{G,k}\, 2^{-2R}, \qquad \alpha_{G,k} = \begin{cases} 2\pi\left(\frac{k+2}{k}\right)^{(k+2)/2}|K|^{1/k}\frac{1}{\sigma_X^2} & \text{for FRC} \\ 2\pi e\,|K|^{1/k}\frac{1}{\sigma_X^2} & \text{for VRC} \end{cases}$$

SNR Gain of Optimal k-dim VQ over Opt k-dim'l Transform Coding

$$10\log_{10}\frac{\delta_{tr}(k,R)}{\delta_{vq}(k,R)} \cong 10\log_{10}\frac{1/12}{m_k^*}\,\frac{|K_{\underline{X}}|^{1/k}\alpha_{G,1}}{\sigma_X^2\,\alpha_{G,k}} = 10\log_{10}\frac{1/12}{m_k^*} \times \begin{cases} \dfrac{3^{3/2}}{\left(\frac{k+2}{k}\right)^{(k+2)/2}} & \text{for FRC} \\ 1 & \text{for VRC} \end{cases}$$

SNR Gain of Optimal high dim'l VQ over high dim'l Transform Coding in dB

$$10 \log_{10} \frac{\delta_{tr}(\infty,R)}{\delta_{vq}(\infty,R)} \cong 10 \log_{10} \frac{1/12}{1/2\pi e} + 10 \log_{10} \begin{cases} \frac{3^{3/2}}{e} & \text{for FRC} \\ 1 & \text{for VRC} \end{cases}$$

$$= 1.53 + \begin{cases} 2.81 & \text{for FRC} \\ 1 & \text{for VRC} \end{cases} = \begin{cases} 4.35 & \text{for FRC} \\ 1.53 & \text{for VRC} \end{cases}$$

These are the same gains as optimal VQ over optimal SQ for IID Gaussian source.

Why?

When optimized, transform coding suffers no memory loss,
but it suffers the same cubic, oblongitis and point density losses as
optimized scalar/product quantization for the IID case.

FRC: One can show that transform coding could be designed

(a) To have optimal point density. In this case it would have high oblongitis loss.

(b) To have cubic cells. In this case it suffers large point density loss

The optimal is a compromise that causes same losses as in the IID case.

VRC: One can design the transform code to have the optimal point density (which is uniform) and cubic cells. So it suffers only the cubic loss.

THE EFFECT OF DIMENSION k ON $|K^{(k)}|$

Let $K^{(k)}$ be the k×k covariance matrix of X with eigenvalues $\lambda_1^{(k)},\ldots,\lambda_k^{(k)}$.

Fact 10: $|K^{(k)}| = M_{k-1}\,|K^{(k-1)}| = \sigma_X^2 \prod_{i=1}^{k-1} M_i$

where M_k is the MSE of the best linear predictor for X_i from $X_{i-k},\ldots,X_{i-1}$.

Proof: Will be given later when we discuss DPCM.

Fact 11: $|K^{(k+1)}|^{1/(k+1)} \leq |K^{(k)}|^{1/k}$

This implies that increasing the dimension of transform coding will not decrease its performance.

Proof: By Fact 10,

$|K^{(k)}|^{1/k}$ = geometric average of $\sigma_X^2, M_1, \ldots, M_k$

$|K^{(k+1)}|^{1/(k+1)}$ = geometric average of $\sigma_X^2, M_1, \ldots, M_k, M_{k+1}$.

Observation: $M_{k+1} \leq M_k$, because the best (k+1)th order predictor must be at least as good as the best kth order predictor.

Observation: the second geometric aveage is like the first except it has one additional term that is no larger than all the others.

Therefore, the second geometric average is no larger than the first.

Fact 12: $\lim_{k\to\infty} |K^{(k)}|^{1/k} = Q$ (stated earlier without proof)

where

$$Q = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln S(\omega)\, d\omega \right\} = \text{"one-step prediction error"}$$

= MSE of optimum linear predictor for X_i based on $X_{i-1}, X_{i-2}, \ldots$

$$S(\omega) = \sum_{n=-\infty}^{\infty} R_X(n)\, e^{-jn\omega}$$ = power spectral density of random process X.

Proof:
$$\lim_{k\to\infty} |K^{(k)}|^{1/k} = \lim_{k\to\infty} \left(\prod_{i=1}^{k} \lambda_i^{(k)}\right)^{1/k}$$
$$= \lim_{k\to\infty} \exp\left\{ \ln \left(\prod_{i=1}^{k} \lambda_i^{(k)}\right)^{1/k} \right\}$$
$$= \lim_{k\to\infty} \exp\left\{ \frac{1}{k} \sum_{i=1}^{k} \ln \lambda_i^{(k)} \right\}$$
$$= \exp\left\{ \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \ln \lambda_i^{(k)} \right\}$$

To complete the proof we need to show

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \ln \lambda_i^{(k)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln S(\omega)\, d\omega$$

Fact 13: **Szego's Eigenvalue Distribution Theorem**

Let $\{X_i\}$ be wide-sense stationary random process with power spectral density $S(\omega)$ and with k-dimensional covariance matrix $K^{(k)}$ having eigenvalues $\lambda_1^{(k)},\ldots,\lambda^{(k)}$. Then for any piecewise continuous function g

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} g(\lambda_i^{(k)}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(S(\omega))\, d\omega$$

References:

U. Grenander and G. Szego, Toeplitz Forms and Their Applications (book)

R.M. Gray, Toeplitz and Circulant Matrices: A Review , (paper) http://www-ee.stanford.edu/~gray/toeplitz.html

Interpretation: This theorem determines the asymptotic "distribution" of the eigenvalues of the covariance matrices of $\{X_i\}$, e.g. when k is large, it determines what fraction of the eigenvalues lie between a and b for any a,b. See next page.

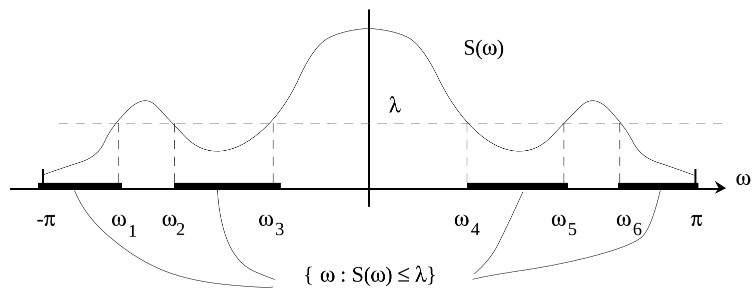
Let $g_\lambda(s) = \begin{cases} 1, & s \le \lambda \\ 0, & \text{else} \end{cases}$

$$F(\lambda) = \lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} g_\lambda(\lambda_i^{(k)}) = \text{asympt frac of e.v.'s} \le \lambda$$

$$= \text{distribution functon for the eigenvalues}$$

The Theorem shows

$$F(\lambda) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g_\lambda(S(\omega))\, d\omega = \frac{\text{length of } \{\omega : S(\omega) \le s\}}{2\pi}$$



Therefore the "density" of eigenvalues with values near s is

$$f(\lambda) = \frac{d}{d\lambda} F(\lambda) \cong \frac{1}{2\pi}\left(\frac{1}{|S'(\omega_1)|} + \frac{1}{|S'(\omega_2)|} + \dots \right)$$

where ω_1, ω_2, … are the frequencies such that $S(\omega_1) = \lambda$.
We see that there are many eigenvalues where slope is flat and few where slope is steep.

Completion of proof of Fact 12:

Let $g(s) = \ln(s)$.

Then by Fact 13 (the eigenvalue distribution theorem)

$$\lim_{k\to\infty} |K^{(k)}|^{1/k} = \exp\left\{\lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} \ln \lambda_i^{(k)} \right\} = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln S(\omega)\, d\omega\right\},$$

which is the desired result.

EXAMPLE: FIRST-ORDER AR, GAUSSIAN SOURCE:

- $X_i = \rho X_{i-1} + Z_i$

  where Z_i's are IID Gaussian, zero mean with variances σ_Z^2, and Z_i is independent of past X's. In this case,

- Best linear predictor for X_i from $X_{i-1},\ldots,X_{i-k}$ is $\tilde{X}_i = X_{i-1}$. Its MSE is σ_Z^2.

$$\Rightarrow \quad M_1 = M_2 = M_3 = \ldots = \sigma_Z^2 = \sigma_X^2(1-\rho^2) = Q = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln S(\omega)\,d\omega\right\}$$

- Fact 10: $\Rightarrow |K^{(k)}| = M_{k-1}|K^{(k-1)}| = \sigma_X^2 \prod_{i=1}^{k-1} M_i = \sigma_X^{2k}(1-\rho^2)^{k-1}$

$$\Rightarrow |K^{(k)}|^{1/k} = \sigma_X^2(1-\rho^2)^{(k-1)/k}$$ (this fact was just stated in Zador sections)

$$\rightarrow \sigma_X^2(1-\rho^2) \text{ as } k \rightarrow \infty.$$

This is larger than $M_1 = \sigma_X^2(1-a^2)$, but converges to it as $k \rightarrow \infty$.

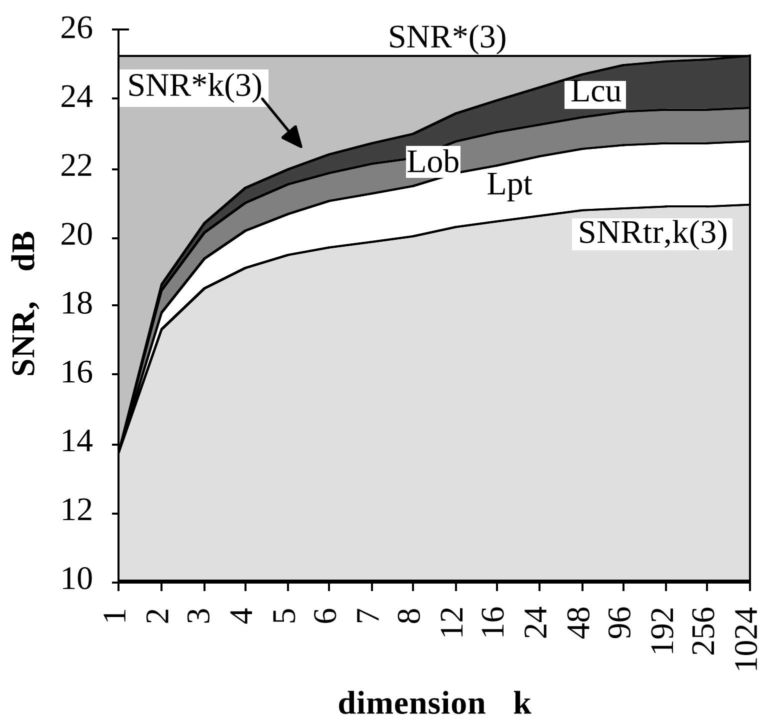
- OPTA, k-dim. Transf. Coding of Stat'ry, Gaussm AR Source with corr. coef. ρ: For large R,

$$\delta_{tr}(k,R) \cong \frac{1}{12}\sigma_X^2(1-\rho^2)^{(k-1)/k}\alpha_{G,1}\,2^{-2R}, \qquad \alpha_{G,1} = \begin{cases} 2\pi 3^{3/2} = 32.6 & \text{for FRC} \\ 2\pi e = 17.08 & \text{for VRC} \end{cases}$$

$$\rightarrow \frac{1}{12}\sigma_X^2(1-\rho^2)\alpha_{G,1}\,2^{-2R} \text{ as } k \rightarrow \infty$$

SNR FOR FIXED-RATE TRANSFORM CODING

Gauss AR Source -- corr. coeff. ρ = .9, R = 3



(Ignore the losses. They are defined in a different way than before.)